

# On the Existence of Equilibrium with Incomplete Markets and Non-monotonic Preferences\*

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## Abstract

We provide a shorter proof than Geanakoplos and Polemarchakis (1986) of the existence of equilibrium in an incomplete financial market economy with numeraire assets, under the weak assumption that asset returns are non-negative. Furthermore, we relax the strict monotonicity assumption on preferences and as an application we prove the existence of equilibrium when agents may disagree on zero probability events but do not plan to go bankrupt in any state.

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### 1. Introduction

We prove a version of the theorem of Geanakoplos and Polemarchakis (1986) on the existence of equilibrium in two-period exchange economies with incomplete, numeraire asset markets. By assuming non-negative asset return, we provide a shorter proof and, moreover, we relax the monotonicity assumption. In particular, one single agent with strictly monotone preferences would suffice. As an application, we show that with von Neumann-Morgenstern utility functions, even if agents disagree on zero-probability events but do not plan to go bankrupt in any state, equilibrium still exists. Therefore, even with incomplete markets, in a two-period model with no bankruptcy, non-equivalent beliefs pose no problem for the existence of equilibrium.

### 2. The Model

The model is similar to that developed by Geanakoplos and Polemarchakis (1986).  $\mathcal{H} = \{0, 1, \dots, H\}$ , with  $H \in \mathbb{N}$ , is a set of agents.  $\mathcal{S} = \{0, 1, \dots, S\}$ , with  $S \in \mathbb{N}$ , is the set of possible future states of the world.  $\mathcal{L} = \{0, 1, \dots, L\}$ , with  $L \in \mathbb{N}$ , is the set of commodities, so the consumption set is  $\mathbb{R}_+^{(L+1)(S+1)}$ . Each agent  $h \in \mathcal{H}$  is endowed with  $w^h \in \mathbb{R}_+^{(L+1)(S+1)}$ .  $\mathcal{A} = \{0, 1, \dots, A\}$ , with  $A \in \mathbb{N}$ , is the set of assets. The return of asset  $a \in \mathcal{A}$  is the (column) vector  $v^a \in \mathbb{R}^S$  which pays in units of the first commodity,  $l = 0$  (the numeraire of the economy). The matrix of asset returns is  $V = [ v^0 \ v^1 \ \dots \ v^A ]$ . For each  $s \in \mathcal{S}$ , denote  $v_s = (v_s^0, v_s^1, \dots, v_s^A) \in \mathbb{R}^{A+1}$ , taken as a row vector. A portfolio is a (column) vector  $y \in \mathbb{R}^{A+1}$ . Let  $p \in \mathbb{R}_+^{(L+1)(S+1)}$  and  $q \in \mathbb{R}^{(A+1)}$  denote commodity spot prices and asset prices, respectively.

Agent  $h$  has preferences represented by utility function  $W^h : \mathbb{R}_+^{(L+1)(S+1)} \rightarrow \mathbb{R}$  and, at prices  $(q, p)$ , faces the budget set

$$B^h(q, p) = \left\{ (y, x) \in \mathbb{R}^A \times \mathbb{R}_+^{(L+1)(S+1)} : \begin{array}{l} q \cdot y = 0 \\ \forall s \in \mathcal{S}, p_s \cdot (x_s - w_s^h) \leq p_{s,0} (v_s \cdot y) \end{array} \right\}$$

Agents maximize their utilities subject to their budget constraint.

We interpret this model as one in which only assets are traded at time zero ( $t = 0$ ) but consumption plans are made for time one ( $t = 1$ ) in each state of nature that might occur in that period,  $s \in \mathcal{S}$ . At time zero net expenditure in assets is zero and at time one agents receive an endowment in each state of nature that they can sell to finance consumption and net financial positions.

In what follows,  $\mathbf{1}_{s,l}$  is a vector in  $\mathbb{R}^{(L+1)(S+1)}$  that contains 0 everywhere, except in the component  $(s, l)$ , where it is 1.

We will use the following assumptions.

**Condition 1** For each  $h$ ,  $W^h$  is continuous and quasiconcave on  $\mathbb{R}_+^{(S+1)(L+1)}$ .

**Condition 2** For each  $h$ ,  $w^h \gg 0$ .

**Condition 3**  $V > 0$  (i.e.,  $V \geq 0, V \neq 0$ ).

**Condition 4**  $\forall s \in \mathcal{S}, \exists h \in \mathcal{H}$  such that for all  $x \in \mathbb{R}_+^{(S+1)(L+1)}$  and  $\varepsilon > 0$ ,  $W^h(x + \varepsilon \mathbf{1}_{s,0}) > W^h(x)$ .

**Remark 1** Condition 4 is considerably weaker than Condition (A3) in Geanakoplos and Polemarchakis (1986): there, Condition (A3) requires that every agent strictly prefer more of the numeraire in each state. Here, Condition 4 requires that, in any state, there be at least one agent that strictly prefers more of the numeraire in that state. In particular, one single agent with strictly monotonic preferences, as in Geanakoplos and Polemarchakis (1986), would imply Condition 4.

For normalized prices of assets and commodities, define  $Q = \{q \in \mathbb{R}_+^{A+1} : \|q\| = 1\}$ ,  $\Delta = \{p \in \mathbb{R}_+^{L+1} : \|p\| = 1\}$ , where, given  $x \in \mathbb{R}^m$ ,  $\|x\| \equiv \sum_{i=1}^m |x_i|$ .

Our equilibrium concept is:

**Definition 1** A competitive equilibrium with a numeraire financial structure  $V$  is a 4-tuple of asset prices, commodity prices, asset allocations and commodity allocations  $(q^*, p^*, y^*, x^*) \in Q \times \Delta^{S+1} \times \mathbb{R}^{(A+1)(H+1)} \times \mathbb{R}_+^{(L+1)(S+1)(H+1)}$  such that:

1. For all  $h \in \mathcal{H}$ ,  $(y^{*h}, x^{*h}) \in \arg \max_{(y,x) \in B^h(q^*, p^*)} W^h$ .
2.  $\sum_{h \in \mathcal{H}} (x^{*h} - w^h) \leq 0$  and, for every  $s$ ,

$$p_{s,l}^* > 0 \Rightarrow \sum_{h \in \mathcal{H}} (x_{s,l}^{*h} - w_{s,l}^h) = 0$$

3.  $\sum_{h \in \mathcal{H}} y^{*h} = 0$ .

**Remark 2** Condition 3 implies little loss of generality, for it suffices that just one asset with strictly positive payoffs (for instance a riskless bond) be available in the economy. That is, consider a financial structure  $\tilde{V} = [\tilde{v}^0 \ \tilde{v}^1 \ \dots \ \tilde{v}^A]$  such that  $\tilde{v}^0 \gg 0$  (but which may violate Condition 3). Find a scalar  $k$  such that  $\tilde{v}^a + k\tilde{v}^0 \geq 0$  for each  $a = 1, \dots, A$ , and construct an alternative financial structure  $V = [v^0 \ v^1 \ \dots \ v^A]$ , with  $v^0 = \tilde{v}^0$  and  $v^a = \tilde{v}^a + k\tilde{v}^0$  for every  $a = 1, \dots, A$ . It is immediate that structure  $V$  satisfies Condition 3, and it follows by construction and definition that the economy has a competitive equilibrium  $(q^*, p^*, x^*, y^*)$  with

structure  $V$ , only if  $(\tilde{q}^*, \tilde{p}^*, \tilde{x}^*, \tilde{y}^*)$  is an equilibrium with structure  $\tilde{V}$ , for  $\tilde{p}^* = p^*$ ,  $\tilde{x}^{*h} = x^{*h}$ ,  $\tilde{q}_0^* = q_0^*$ ,  $\tilde{q}_a^* = q_a^* - kq_0^*$  for each  $a = 1, \dots, A$ , and  $\tilde{y}_0^{*h} = y_0^{*h} + k \sum_{a=1}^A y_a^{*h}$  and  $\tilde{y}_a^{*h} = y_a^{*h}$ , for each  $a = 1, \dots, A$ , for all  $h$ .

For completeness, we include the following.

**Lemma 1** Let  $K \subset \mathbb{R}^{A+1} \times \mathbb{R}^{(S+1)(L+1)}$  be a compact rectangle with center at the origin. For each individual  $h \in \mathcal{H}$ , consider the truncated individual demand correspondence  $\hat{f}^h(\cdot; K) : \mathbb{R}^{A+1} \times \mathbb{R}_+^{(S+1)(L+1)} \rightrightarrows \mathbb{R}^{A+1} \times \mathbb{R}_+^{(S+1)(L+1)}$ , defined by

$$\hat{f}^h(q, p; K) = \arg \max_{(y,x) \in B^h(q,p;K)} W^h$$

where  $B^h(q, p; K) = B^h(q, p) \cap K$ . Under Conditions 1 and 2,  $\hat{f}^h(\cdot; K)$  is nonempty-, compact-, convex-valued and upper hemicontinuous at each  $(q, p) \in \mathbb{R}^{A+1} \times \mathbb{R}_+^{(S+1)(L+1)}$ , with  $q \neq 0$  and  $p_s \neq 0$  for all  $s$ .

**Proof** (Geanakoplos and Polemarchakis, 1986). By continuity and compactness,  $\hat{f}^h(q, p; K)$  is nonempty and compact, and by quasiconcavity of  $W^h$  and the convexity of the budget set, it is convex.

To show upper hemicontinuity at each  $(q, p) \in \mathbb{R}^A \times \mathbb{R}_+^{(S+1)(L+1)}$  with  $q \neq 0$  and  $p_s \neq 0$  for all  $s$ , let  $(q_n, p_n)_{n=1}^\infty$  be a sequence such that  $(q_n, p_n) \rightarrow (q, p)$  and let  $(y_n, x_n)_{n=1}^\infty$  be such that  $(y_n, x_n) \in \hat{f}^h(q_n, p_n; K)$ . Since  $(y_n, x_n)_{n=1}^\infty$  lies in  $K$ , there exists a convergent subsequence  $(y_{n(k)}, x_{n(k)})_{k=1}^\infty \rightarrow (y, x) \in B^h(q, p; K)$ .

Assume that  $(y, x) \notin \hat{f}^h(q, p; K)$ . Then, there exists  $(\bar{y}, \bar{x}) \in B(q, p; K)$  such that  $W^h(\bar{x}) > W^h(x)$ . By continuity, for  $\lambda < 1$  but close enough to 1, we have  $W^h(\lambda \bar{x}) > W^h(x)$ ; and, again by continuity, for large enough  $k$ ,  $W^h(\lambda \bar{x}) > W^h(x_{n(k)})$ .

Let  $\bar{y}_n = \arg \min \{\|\bar{y} - y'\| : y' \cdot q_n = 0\}$ . Then by the Theorem of the Maximum, since  $q \neq 0$ , then  $q_n \rightarrow q$  implies that  $\bar{y}_n \rightarrow \bar{y}$ . We know that  $(\bar{y}, \bar{x}) \in B(q, p; K)$ . Since  $\forall s \in \mathcal{S}$ ,  $p_s \cdot w_s^h \in \mathbb{R}_{++}$ , it is easy to see that for  $\lambda < 1$ , but close to 1,  $p_s \cdot (\lambda \bar{x}_s - w_s^h) < p_{s,0} (v_s \cdot \lambda \bar{y})$  and  $(\lambda \bar{y}, \lambda \bar{x}) \in K$ . Since  $(q_{n(k)}, p_{n(k)}) \rightarrow (q, p)$  and  $\bar{y}_{n(k)} \rightarrow \bar{y}$  then, for large  $k$ , we have  $p_{n(k),s} \cdot (\lambda \bar{x}_s - w_s^h) < p_{n(k),s,0} (v_s \cdot \lambda \bar{y}_{n(k)})$  and  $(\lambda \bar{y}_{n(k)}, \lambda \bar{x}) \in K$  which means that, for large enough  $k$ ,  $(\lambda \bar{y}_{n(k)}, \lambda \bar{x}) \in B(q_{n(k)}, p_{n(k)}, w; K)$  and, hence  $W^h(x_{n(k)}) \geq W^h(\lambda \bar{x})$ , which is impossible. ■

**Theorem 1** Under Conditions 1 – 4, there exists a competitive equilibrium.

**Proof** Assume, without loss of generality, that  $V$  has full column rank.

Fix  $n \in \mathbb{N}$ , and let

$$K_n^h = \left[ -\frac{n}{H+1}, \frac{n}{H+1} \right]^{A+1} \times \left\{ x \in \mathbb{R}^{(S+1)(L+1)} : -nw^h \leq x \leq nw^h \right\}$$

$$K_n = [-n, n]^{A+1} \times \left\{ x \in \mathbb{R}^{(S+1)(L+1)} : -n \sum_{h \in \mathcal{H}} w^h \leq x \leq n \sum_{h \in \mathcal{H}} w^h \right\}$$

and let  $\widehat{F}(\cdot; n) : \mathbb{R}^{A+1} \times \mathbb{R}_+^{(S+1)(L+1)} \rightrightarrows \mathbb{R}^{A+1} \times \mathbb{R}_+^{(S+1)(L+1)}$  be the aggregate truncated demand correspondence, defined by  $\widehat{F}(q, p; n) = \sum_{h \in \mathcal{H}} \widehat{f}^h(q, p; K_n^h)$ .

Then  $\widehat{F}(\cdot; n)$  is nonempty-, compact-, convex-valued and upper hemicontinuous at each  $(q, p) \in \mathbb{R}^A \times \mathbb{R}_+^{(S+1)(L+1)}$ ,  $q \neq 0$  and  $p_s \neq 0$  for all  $s$ . Also, if  $(y^h, x^h) \in \widehat{f}^h(q, p; K_n^h)$ , then  $p_s \cdot (x_s^h - w_s^h) \leq p_{s,0} (v_s \cdot y^h)$ , so if  $(y, x) \in \widehat{F}(q, p; n)$ , then  $p_s \cdot (x_s - \sum_{h \in \mathcal{H}} w_s^h) \leq p_{s,0} (v_s \cdot y)$ .

Define correspondence  $\Phi : Q \times \Delta^{S+1} \times K_n \rightrightarrows Q \times \Delta^{S+1} \times K_n$ , by  $\Phi_1 \times \Phi_2 \times \Phi_3$ , where

$$\Phi_1(q, p, (y, z)) = \arg \max_{q' \in Q} \{q' \cdot y\} \subseteq Q$$

$$\Phi_2(q, p, (y, z)) = \prod_{s \in \mathcal{S}} \arg \max_{p'_s \in \Delta} \{p'_s \cdot z_s\} \subseteq \Delta^{S+1}$$

$$\Phi_3(q, p, (y, z)) = \widehat{F}(q, p; n) - \left( 0, \sum_{h \in \mathcal{H}} w^h \right) \subseteq K_n$$

$\Phi_1$  and  $\Phi_2$  are nonempty-, compact-, convex-valued and upper hemicontinuous and, by Lemma 1,  $\Phi_3$  has the same properties. Therefore,  $\Phi$  is nonempty-, compact-, convex-valued and upper hemicontinuous.

By Kakutani's fixed-point theorem, there exists  $(q^*, p^*, y^*, z^*) \in \Phi(q^*, p^*, y^*, z^*)$  (in particular,  $q^* \cdot y^* = 0$ ).

We first note that  $Vy^* \leq 0$ . Suppose that  $v_s \cdot y^* > 0$ ; then, using Condition 3, let  $q = (q^* + v_s) \in \mathbb{R}_+^{A+1} \setminus \{(0, \dots, 0)\}$ , which implies that for some  $\lambda > 0$ ,  $\lambda q \in Q$  and  $\lambda q \cdot y^* = \lambda(q^* + v_s) \cdot y^* = \lambda v_s \cdot y^* > 0$  in contradiction with the fact that  $q^* \in \arg \max_{q' \in Q} \{q' \cdot y^*\}$ .

Secondly,  $p_s^* \cdot z_s^* \leq p_{s,0}^* v_s \cdot y^* \leq 0$ , and, hence,  $z_s^* \leq 0$ . Otherwise, if for some  $l$ ,  $z_{s,l}^* > 0$ , then for  $p_s = (0, \dots, 1, \dots, 0)$ , one would have  $p_s \cdot z_s^* = z_{s,l}^* > 0$ , in contradiction with the fact that  $p_s^* \in \arg \max_{p'_s \in \Delta} \{p'_s \cdot z_s^*\}$  and  $p_s^* \cdot z_s^* \leq 0$ .

Also, since  $z_s^* = \sum_{h \in \mathcal{H}} x_s^{*h} - \sum_{h \in \mathcal{H}} w_s^h \leq 0$  and  $x^{*h} \in \mathbb{R}_+^{(L+1)(S+1)}$ , then  $\|x_s^{*h}\| \leq \left\| \sum_{h' \in \mathcal{H}} w_s^{h'} \right\|$  for all  $h$ .

Now, for each  $n \in \mathbb{N}$ , let  $(q_n^*, p_n^*, y_n^*, z_n^*)$  be a fixed point, and let

$$(y_n^*, z_n^*) = \left( \sum_{h \in \mathcal{H}} y_n^{*h}, \sum_{h \in \mathcal{H}} x_n^{*h} - \sum_{h \in \mathcal{H}} w^h \right)$$

By the above boundedness of  $(x_n^{*h})_{n=1}^\infty$  and the compactness of  $Q$  and  $\Delta^{S+1}$ , there exists a subsequence such that  $(q_{n(k)}^*, p_{n(k)}^*, x_{n(k)}^*) \rightarrow (q^*, p^*, x^*)$  (in particular,  $z_{n(k)}^* \rightarrow z^* = x^* - \sum_{h \in \mathcal{H}} w^h$ , and  $p_s^* \in \arg \max_{p'_s \in \Delta} \{p'_s \cdot z_s^*\}$ ).

Suppose that for some  $s$ ,  $p_{s,0}^* = 0$ . By Condition 4, there is an agent  $h$  such that for every  $\varepsilon > 0$ ,  $W^h(x^{*h} + \varepsilon \mathbf{1}_{s,0}) > W^h(x^{*h})$ . Let  $\varepsilon = \min_l \{w_{s,l}^h\} > 0$  (by Condition 2). Then,  $W^h(x^{*h} + \varepsilon \mathbf{1}_{s,0}) > W^h(x^{*h})$ , so, by continuity, for large  $k$ ,

$$W^h \left( \left(1 - (p_{s,0}^*)_{n(k)}\right) x_{n(k)}^{*h} + \varepsilon \mathbf{1}_{s,0} \right) > W^h \left( x_{n(k)}^{*h} \right)$$

Since  $x_{n(k)}^{*h}$  is bounded and  $(p_{s,0}^*)_{n(k)} \rightarrow 0$ , for large  $k$ ,  $(p_{s,0}^*)_{n(k)} < 1$ , and

$$\left( \left(1 - (p_{s,0}^*)_{n(k)}\right) y_{n(k)}^{*h}, \left(1 - (p_{s,0}^*)_{n(k)}\right) x_{n(k)}^{*h} + \varepsilon \mathbf{1}_{s,0} \right) \in B^h \left( q_{n(k)}^*, p_{n(k)}^*; K_{n(k)} \right)$$

which contradicts the fact that  $(y_{n(k)}^{*h}, x_{n(k)}^{*h}) \in \hat{f}^h \left( q_{n(k)}^*, p_{n(k)}^*; K_{n(k)} \right)$ .

It follows that  $p_{s,0}^* > 0$  and, since  $V$  has full rank, we can define  $y^{*h}$  as the unique solution to

$$v_s \cdot y^{*h} = \frac{1}{p_{s,0}^*} \left( p_s^* \cdot (x_s^{*h} - w_s^h) \right), \text{ for all } s \in S \tag{1}$$

Now, for large  $n$ ,  $(y^{*h}, x^{*h})$  is interior to  $K_n^h$ , and by continuity and quasiconcavity of  $W^h$ ,  $(y^{*h}, x^{*h})$  is maximal in  $B(q^*, p^*)$ . Also,  $q \cdot y^* = 0$  and  $Vy^* \leq 0$ . Now, suppose  $y^* \neq 0$ ; then,  $V(-y^*) \geq 0$  and there is  $s \in S$  such that  $v_s(-y^*) > 0$ ; by Condition 4,  $\exists h \in \mathcal{H}$  such that

$$W^h(x^{*h} + v_s(-y^*) \mathbf{1}_{s,0}) > W^h(x^{*h})$$

whereas  $(y^{*h} - y^*, x^{*h} + v_s(-y^*) \mathbf{1}_{s,0}) \in B^h(p^*, q^*)$ , contradicting the maximality of  $(y^{*h}, x^{*h})$ . Therefore,  $y^* = 0$ .

Finally, by definition of  $y^{*h}$  (see Equation 1),  $p_s^* \cdot (x_s^{*h} - w_s^h) = p_{s,0}^* v_s y^{*h}$ , so  $p_s^* \cdot z_s^* = p_{s,0}^* v_s y^* = 0$ . Hence, since  $z_s^* \leq 0$ , then  $z_{s,l}^* = 0$  whenever  $p_{s,l} > 0$ . ■

### 3. Application

We consider the case in which each agent's utility function has a von Neumann-Morgenstern expected utility representation, but in which agents may disagree on zero-probability events. We carry on the assumption that even if agents assign probability zero to a particular event, they do not plan to go bankrupt on that event.

**Condition 5** For each  $h$ ,  $W^h$  can be written as  $\sum_{s=0}^S \pi^h(s) u_s^h(x_s)$  where  $\pi^h : \mathcal{S} \rightarrow [0, 1]$  is agent  $h$  probability distribution (i.e., beliefs) over  $\mathcal{S}$ ,  $u_s^h$  is continuous, concave.

**Condition 6** For each  $s \in \mathcal{S}$ , there exists an  $h \in \mathcal{H}$  such that, for all  $x \in \mathbb{R}_+^{(S+1)(L+1)}$  and  $\varepsilon > 0$ ,  $u_s^h(x + \varepsilon \mathbf{1}_{s,0}) > u_s^h(x)$ .

**Theorem 2** Under Conditions 2, 3, 5 and 6, there is a 4-tuple of asset prices, commodity prices, asset allocations and commodity allocations  $(q^*, p^*, y^*, x^*) \in Q \times \Delta^{S+1} \times \mathbb{R}^{(A+1)(H+1)} \times \mathbb{R}_+^{(L+1)(S+1)(H+1)}$  such that:

1. For all  $h \in \mathcal{H}$ ,  $(y^{*h}, x^{*h}) \in \arg \max_{(y,x) \in B^h(q^*, p^*)} W^h$  and moreover, for every  $s$ ,

$$x_s^{*h} \in \arg \max_{\{x \in \mathbb{R}_+^{L+1} | p_s^* \cdot x \leq p_s^* \cdot w_s^h + p_{s,0}^* v_s y^{*h}\}} u_s^h$$

2.  $\sum_{h \in \mathcal{H}} (x^{*h} - w^h) \leq 0$  and,

$$p_{s,l} > 0 \Rightarrow \sum_{h \in \mathcal{H}} (x_{s,l}^{*h} - w_{s,l}^h) = 0$$

3.  $\sum_{h \in \mathcal{H}} y^{*h} = 0$ .

Notice that Conditions 5 and 6 allow for beliefs to be completely different. In particular, we do not require them to be equivalent in the sense that for all  $h, j \in \mathcal{H}$ ,  $\pi^h(s) = 0 \Leftrightarrow \pi^j(s) = 0$ . It follows that equivalence of beliefs is not a necessary condition for existence of equilibrium in a two-period economy with no bankruptcy. Also, the theorem requires that  $(p_s^*, x_s^*)$  be a spot market competitive equilibrium in state  $s$ . While this is obvious for states to which all households attach nonzero probability, it has to be argued independently when that is not the case.

**Proof** Assume, without loss of generality, that  $V$  has full column rank.

For each  $n \in \mathbb{N}$ , define the probability measures  $\pi_n^h : \mathcal{S} \rightarrow (0, 1]$  by

$$\pi_n^h(s) = \left( \pi^h(s) + \frac{1}{n} \right) \left( \frac{n}{n + S + 1} \right)$$

and define  $W_n^h : \mathbb{R}_+^{(S+1)(L+1)} \rightarrow \mathbb{R}$  by,  $W_n^h(x) = \sum_{s=0}^S \pi_n^h(s) u_s^h(x_s)$ . Conditions 2, 3, 5 and 6 on  $W_n^h$  imply Conditions 1 – 4. Therefore, by theorem 1, there exists  $(q_n^*, p_n^*, y_n^*, x_n^*) \in Q \times \Delta^{S+1} \times \mathbb{R}^{(A+1)(H+1)} \times \mathbb{R}_+^{(L+1)(S+1)(H+1)}$  such that for all  $h \in \mathcal{H}$ ,

$$(y_n^{*h}, x_n^{*h}) \in \arg \max_{(y,x) \in B^h(q_n^*, p_n^*)} \sum_{s \in \mathcal{S}} \pi_n^h(s) u_s^h(x_s)$$

and for each  $s$ ,

$$x_{n,s}^{*h} \in \arg \max_{\{x \in \mathbb{R}_+^{L+1} | p_{n,s}^* \cdot x \leq p_{n,s}^* \cdot w_s^h + p_{n,s,0}^* (v_s \cdot y_n^{*h})\}} u_s^h(x)$$

and

$$p_{n,s}^* \cdot x_{n,s}^{*h} = p_{n,s}^* \cdot w_s^h + p_{n,s,0}^* (v_s \cdot y_n^{*h}) \tag{2}$$

Also,

$$\sum_{h \in \mathcal{H}} (x_n^{*h} - w^h) \leq 0 \tag{3}$$

$$p_{n,s,l}^* > 0 \Rightarrow \sum_{h \in \mathcal{H}} (x_{n,s,l}^{*h} - w_{s,l}^h) = 0$$

and

$$\sum_{h \in \mathcal{H}} y_n^{*h} = 0$$

By Equation 3, for each  $h \in \mathcal{H}$ ,  $(x_n^{*h})_{n \in \mathbb{N}}$  is bounded, and by Equation 2 it follows that  $(y_n^{*h})_{n \in \mathbb{N}}$  is bounded. Therefore, since  $(q_n^*, p_n^*)$  is also bounded, it follows that there exist  $(q^*, p^*, y^*, x^*) \in Q \times \Delta^{S+1} \times \mathbb{R}^{(A+1)(H+1)} \times \mathbb{R}_+^{(L+1)(S+1)(H+1)}$  and a subsequence such that  $(q_{n(k)}^*, p_{n(k)}^*) \rightarrow (q^*, p^*)$ , and for every  $h, (y_{n(k)}^{*h}, x_{n(k)}^{*h}) \rightarrow (y^{*h}, x^{*h})$ .

Clearly,  $(y^{*h}, x^{*h}) \in B^h(q^*, p^*)$ . We show that  $(y^{*h}, x^{*h}) \in \arg \max_{(y,x) \in B^h(q^*, p^*)} W^h$ . Suppose to the contrary that there exist an  $h \in \mathcal{H}$  and  $(\bar{y}, \bar{x}) \in B^h(q^*, p^*)$  such that  $W^h(\bar{x}) > W^h(x^{*h})$ . By continuity, for  $\lambda < 1$  but close enough to 1, we have  $W^h(\lambda \bar{x}) > W^h(x^{*h})$ . Let  $\bar{y}_n = \arg \min \{\|\bar{y} - y'\| : y' \cdot q_n = 0\}$ . Then by the Theorem of the Maximum, since



$q^* \neq 0$  then  $q_{n(k)} \rightarrow q^*$  implies that  $\bar{y}_{n(k)} \rightarrow \bar{y}$ . We know that  $(\bar{y}, \bar{x}) \in B^h(q^*, p^*)$ . Since  $\forall s \in \mathcal{S}, p_s \cdot w_s^h \in \mathbb{R}_{++}$ , it is easy to see that for  $\lambda < 1$ , but close to 1,  $p_s \cdot (\lambda \bar{x}_s - w_s^h) < p_{s,0}(v_s \cdot \lambda \bar{y})$ . Since  $(q_{n(k)}, p_{n(k)}) \rightarrow (q^*, p^*)$  and  $\bar{y}_{n(k)} \rightarrow \bar{y}$  then, for large  $k$ , we have  $p_{n(k),s} \cdot (\lambda \bar{x}_s - w_s^h) < p_{n(k),s,0}(v_s \cdot \lambda \bar{y}_{n(k)})$  which means that, for  $k$  large enough,  $(\lambda \bar{y}_{n(k)}, \lambda \bar{x}) \in B(q_{n(k)}, p_{n(k)}, w)$  and, hence  $W_{n(k)}^h(x_{n(k)}) \geq W_{n(k)}^h(\lambda \bar{x})$ , which implies  $W^h(x^{*h}) \geq W^h(\lambda \bar{x})$  because  $\pi_n^h \rightarrow \pi^h$ , a contradiction.

Again by the Theorem of the Maximum, and since

$$\{x \in \mathbb{R}_+^{L+1} \mid p_s \cdot x \leq p_s \cdot w_s^h + p_{s,0}(v_s \cdot y)\}$$

defines a continuous correspondence (on  $p_s$  and  $y$ ) at every  $p_s \neq 0$ , it follows that  $(x_{n(k)}^{*h})$  has a subsequence that converges to some point in:

$$\arg \max_{\{x \in \mathbb{R}_+^{L+1} \mid p_s^* \cdot x \leq p_s^* \cdot w_s^h + p_{s,0}^*(v_s \cdot y^{*h})\}} u_s^h$$

But since  $(x_{n(k)}^{*h})_{k \in \mathbb{N}}$  itself converges to  $x_s^{*h}$ , it follows that

$$x_s^{*h} \in \arg \max_{\{x \in \mathbb{R}_+^{L+1} \mid p_s^* \cdot x \leq p_s^* \cdot w_s^h + p_{s,0}^*(v_s \cdot y^{*h})\}} u_s^h$$

If  $p_{s,l}^* > 0$ , then, by construction,  $\sum_{h \in \mathcal{H}} (x_{n(k),s,l}^{*h} - w_{s,l}^h) = 0$ , which implies that  $\sum_{h \in \mathcal{H}} (x_{s,l}^{*h} - w_{s,l}^h) = 0$ . ■

**References**

Geanakoplos, J. & Polemarchakis, H. (1986). *Existence Regularity and Constrained Sub-Optimality of Competitive Allocations When Asset Market is Incomplete. Uncertainty, Information and Communication Essays in Honor of Keneth J. Arrow*, volume III. Cambridge University Press, Cambridge.