The Trade-Off Between Incentives and Endogenous Risk*

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Abstract

Negative relationship between risk and incentives, predicted by standard moral hazard models, has not been confirmed by empirical work. We propose a moral hazard model in which heterogeneous risk-averse agents can control the mean and variance of the profits. The possibility of risk reduction allows the agent's marginal utility of incentives to be increasing or decreasing in risk aversion. Positive relationship between endogenous risk and incentives is found when marginal utility of incentives and variance are decreasing in risk aversion. Exogenous risk and incentives are negatively related. Those results remain when adverse selection precedes moral hazard.

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1. Introduction

Moral hazard plays a central role in problems involving delegation of tasks. When the principal cannot perfectly observe the effort exerted by a risk-averse agent, the payment must be designed by taking into account the trade-off between incentives and risk sharing. As the optimal level of incentives depends on the variance of output, the relationship between risk and incentives is an important testable implication of incentive models.

Standard models of moral hazard predict a negative relationship between risk and incentives. The central reference is the model presented in Holmstrom and Milgrom (1987). They analyze the conditions in which optimal contracts are linear, that is, the agent's payoff is a fixed part plus a proportion of profits. In their model, the negative relationship between risk and incentives results from the interaction between these two variables in the agent's risk premium. As the agent is risk averse and incentives bring risk to the agent's payoff, incentives incur a cost in utility. At the optimal incentive, an increase in risk is balanced by a reduction in incentives.

The empirical work does not verify the negative relationship between risk and incentives, and sometimes it finds opposite results. Prendergast (2002) presents a survey of empirical studies in three fields of application, namely, executive compensation, sharecropping and franchising. Positive or insignificant relationships are found in all three fields, while negative relationship is found only in studies on executive compensation. The conclusion is that the evidence is weak. Similarly, in the insurance literature, the monotone relationship between risk and coverage is not verified, as reported in Chiappori and Salanié (2000).

The lack of empirical support has stimulated the search for alternative models that are compatible with the observed facts. Prendergast (2002) suggests a theoretical model that assumes that monitoring is harder in riskier environments. As incentives are a substitute for monitoring, incentives and risk are positively related. His model departs from the Holmstrom–Milgrom structure and risk aversion plays no role. To analyze contracts in agriculture, Ghatak and Pandey (2000) develop a moral hazard model assuming linear contracts, risk-neutral agents and limited liability. Their model is related to ours, since the agent controls the mean and variance of output. However, as limited liability induces riskier behavior, they obtain the optimization trade-off by assuming that the agent pays a cost to *increase* the risk of the project.

We propose a moral hazard model with risk-neutral principal and risk-averse agent who can control the mean and the variance of the profits.[1] The linearity result of Holmstrom and Milgrom (1987) is extended in Sung (1995), which shows that linear contracts are optimal in continuous time moral hazard problems with

[1] Models in which the agent exerts effort in multiple activities were developed in Holmstrom and Milgrom (1991), but in these models, effort controls exclusively the mean of the profits.

controllable variance. We have to distinguish two types of risk: the exogenous or natural risk is the risk of the project when the agent exerts no effort in variance reduction, and the endogenous risk is the one resulting from the agent's effort in variance reduction. Ideally, empirical tests should use the former, but, as it is very difficult to measure, measures of risk used in practice are endogenous.[2] We show that if we allow for heterogeneity in risk aversion, positive relationship between endogenous risk and incentives is possible. This situation occurs when more risk-averse agents receive lower incentive contracts but exert more effort in risk reduction. We examined the linear-quadratic specification for the model and mapped the values of parameters that generate positive or negative relationship between endogenous risk and incentives.[3] The positive relationship is more likely for low risk aversion and intermediate levels of exogenous risk. We also found that the exogenous risk and incentives remain negatively related for usual functional specifications.

We also examine an extension with adverse selection before moral hazard. In this case, the principal does not observe the agent's risk aversion and designs a menu of contracts so that self-selection reveals the type of agent. Sung (2005) shows that linear contracts are optimal for mixed models of adverse selection before moral hazard and controllable variance. We computed the optimal contracts for representative situations and found that the relationship between endogenous risk and incentives is ambiguous. For a set of agent types with high risk aversion, incentives and endogenous risk are negatively related. Conversely, for a set of agents with low risk aversion, the relationship is positive. With respect to the exogenous risk, the Holmstrom–Milgrom result is preserved: exogenous risk and incentives are negatively related.[4] Sung (2005) also uses a linear-quadratic specification and obtains a positive relationship between risk and incentive in the adverse selection with moral hazard version of the model, but not in the pure moral hazard version. That model assumes that the agents are heterogeneous with respect to an efficiency parameter in the cost function and requires that the marginal effect of the effort on profits be dependent on the level of risk. In our model, the positive relationship both in pure moral hazard and in adverse selection with moral hazard versions is obtained only with the assumptions of controllable variance and heterogeneity in risk aversion.

The rest of the paper is organized as follows. In Section 2, we present the moral hazard model with controllable variance and analyze the relationship between risk and incentives in the linear-quadratic specification. In Section 3, as a robustness check, we examine the case in which the agent's risk aversion is not observable. We

[2] Allen and Lueck (1999) and Lafontaine and Bhattacharyya (1995) discuss the difficulty in measuring the exogenous risk.

[3] In our linear-quadratic specification, the mean of profits is linear, and the variance and the cost functions are quadratic functions of efforts.

[4] In Araujo and Moreira (2001a), a model with adverse selection and moral hazard is applied to the insurance market and an ambiguous relationship between coverage and risk is found.

present the adverse selection with moral hazard model and compute relevant cases of optimal contracts in the linear-quadratic specification. We find positive and negative relationships between risk and incentives. Section 4 states the concluding remarks; in the Appendix, we present the proofs of the propositions and discuss implementability and optimality in adverse selection models without the single-crossing property.

2. Observable Risk Aversion

The principal delegates the management of the firm to the agent, whose effort can affect the probability distribution of the profits.[5] The agent may exert effort e increasing the mean, and effort f reducing the variance. The profits, denoted by z, have normal distribution $N(\mu(e), \sigma^2(f))$. For simplicity, we assume that the cost function is separable into two components denoted as $c(e)$ and $k(f)$. The agent has exponential utility with risk aversion θ, which is publicly known. As shown in Sung (1995), linear contract is optimal in the Holmstrom–Milgrom setting where agent has control of risk. So the wage is a linear function of the profits, that is, $w = \alpha z + \beta$, $\alpha \geq 0$. The contract parameter α is the proportion of the profits received by the agent and is called the incentive, or the power, of the contract. The parameter β is the fixed part of the contract which is adjusted in order to induce the agent to participate.

The timing of the problem is as follows: (1) the principal and the agent learn the type θ, then (2) the principal offers a contract $w = \alpha z + \beta$, (3) the agent may accept or reject the contract. If he accepts it, then (4) he exerts effort accordingly, (5) the firm produces profit z, (6) the agent receives $w = \alpha z + \beta$ and the principal earns the net profit, $z - w$. The certainty equivalence of the agent's utility is

$$V_{CE}(\alpha, \beta, \theta, e, f) = \beta + \alpha\mu(e) - c(e) - k(f) - \frac{\alpha^2}{2}\theta\sigma^2(f)$$

that is, the expected wage, minus the cost of the efforts and the risk premium. In the traditional moral hazard model, the last term originates the negative relationship between risk and incentives. The risk premium acts as a cost because the principal must compensate the agent to induce him to participate. Since the marginal risk premium with respect to α is increasing in both α and σ^2, the principal compensates an increase in σ^2 by reducing α to the level at which the marginal cost and the marginal benefit of incentive are equal. In our model, the last term has the same role, but the possibility of variance reduction modifies the relationship between risk and incentives.

The costs are convex and efforts increase mean and reduce variance with diminishing returns to scale. The following assumption summarizes these properties:

[5] We refer to the principal with feminine pronouns and to the agent with masculine pronouns.

Assumption 1 $c'(\cdot) > 0$, $c''(\cdot) > 0$, $k'(\cdot) > 0$, $k''(\cdot) > 0$, $\mu'(\cdot) > 0$, $\mu''(\cdot) \leq 0$, $\sigma^{2\prime}(\cdot) < 0$ and $\sigma^{2\prime\prime}(\cdot) > 0$.

2.1 Solving the agent's problem

Given the contract (α, β), the agent with risk aversion θ chooses effort levels e^* and f^* that maximize his utility. The first-order conditions for the agent's problem are[6]

$$\alpha\mu'(e^*) = c'(e^*) \qquad \text{and} \qquad -\frac{1}{2}\alpha^2\theta\sigma^{2\prime}(f^*) = k'(f^*) \tag{1}$$

Let $e^*(\alpha)$ and $f^*(\alpha, \theta)$ denote the agent θ's optimal choice of efforts, given the incentives α. Differentiating the first-order condition, we find that the derivatives of effort with respect to incentives and risk aversion have well defined signs,

$$e^*_\alpha = \frac{\mu'(e^*)}{c''(e^*) - \alpha\mu''(e^*)} > 0$$

$$f^*_\alpha = -\frac{\alpha\theta\sigma^{2\prime}(f^*)}{k''(f^*) + \frac{1}{2}\alpha^2\theta\sigma^{2\prime\prime}(f^*)} > 0 \tag{2}$$

$$f^*_\theta = -\frac{\frac{1}{2}\alpha^2\sigma^{2\prime}(f^*)}{k''(f^*) + \frac{1}{2}\alpha^2\theta\sigma^{2\prime\prime}(f^*)} > 0 \tag{3}$$

The higher the incentive, the higher the effort in mean increase and in variance reduction. The higher the risk aversion, the higher the effort in variance reduction. Consequently, the endogenous variance is decreasing in α and in θ. This is the expected result, since higher α provides incentive for the agent to increase average profits, but simultaneously increases the risk of his payoff. The risk-averse agent is induced to reduce risk by increasing f^*, and this effect is stronger, the higher the risk aversion. Using $e^*(\alpha)$ and $f^*(\alpha, \theta)$, the indirect utility is $V(\alpha, \beta, \theta) = \beta + v(\alpha, \theta)$, which is quasi-linear in β. The non-linear term is

$$v(\alpha, \theta) = \alpha\mu(e^*(\alpha)) - c(e^*(\alpha)) - k(f^*(\alpha, \theta)) - \frac{1}{2}\alpha^2\theta\sigma^2(f^*(\alpha, \theta)) \tag{4}$$

By the envelope theorem, we find that $v_\theta(\alpha, \theta) = -\alpha^2\sigma^2(f^*(\alpha, \theta))/2 < 0$. This means that, comparing two agents with marginally distinct risk aversion under the same contract (α, β), the more risk-averse agent has higher risk premium and lower utility, even when his choice of risk-reduction effort is taken into account.

[6] We assume the existence of the solutions. This can be obtained with additional mild conditions, for example, Inada conditions.

2.2 The principal's problem

We assume that the principal is risk-neutral. Her utility is, given the agent's effort choice, the expectation of the net profit,

$$U(\alpha, \beta) = E[z - w] = (1 - \alpha)\mu(e^*(\alpha)) - \beta$$

where the expectation is taken with respect to the conditional distribution of z, given the agent θ's effort choice under the contract (α, β). Denote the optimal contract for the agent of type θ as $(\alpha(\theta), \beta(\theta))$. For all θ, this is the solution of the principal's maximization problem, $\max_{\tilde{\alpha},\tilde{\beta}} U(\tilde{\alpha}, \tilde{\beta})$, subject to the participation constraint $V(\tilde{\alpha}, \tilde{\beta}, \theta) \geq 0$. As it holds with equality, $\beta(\theta)$ may be eliminated from the objective function. The principal's problem is then, for all θ, to choose $\alpha(\theta)$ that maximizes the social surplus, $S(\alpha, \theta) = (1 - \alpha)\mu(e^*(\alpha)) + v(\alpha, \theta)$, whose first derivative with respect to incentives is

$$S_\alpha(\alpha, \theta) = (1 - \alpha)\mu'(e^*(\alpha))e^*_\alpha(\alpha) - \alpha\theta\sigma^2(f^*(\alpha, \theta)) \tag{5}$$

So, the first-order condition, $S_\alpha(\alpha(\theta), \theta) = 0$, implies a balance between the increase in the mean of the profits represented by the first term, and the marginal cost associated with the risk premium represented by the second term. Assuming $\theta > 0$, $S_\alpha(\alpha, \theta)$ is positive for $\alpha < 0$ and negative for $\alpha > 1$. Thus, by continuity, for a given θ, the maximum value of the social surplus is attained for $\alpha(\theta) \in (0, 1)$, and $S_{\alpha\alpha}(\alpha(\theta), \theta) < 0$.

We are interested in the relationship between risk and incentives when the population of agents is heterogeneous with respect to the risk aversion. For an agent with a higher risk aversion coefficient, the principal may assign a contract with more or less incentives, and the agent may exert more or less effort in risk reduction. We show that the positive relationship between risk and incentives is found when incentives decrease and effort in risk reduction increases with risk aversion.

We begin the analysis by investigating the relationship between risk aversion and incentives. As $S_{\alpha\theta}(\alpha, \theta) = v_{\alpha\theta}(\alpha, \theta)$, using the implicit function theorem on the first-order condition,

$$\frac{d\alpha}{d\theta} = -\frac{v_{\alpha\theta}(\alpha(\theta), \theta)}{S_{\alpha\alpha}(\alpha(\theta), \theta)} \tag{6}$$

which states that the relationship between incentives and risk aversion has the same sign as $v_{\alpha\theta}(\alpha(\theta), \theta)$, and reveals a close relationship between the cross-derivative of agent's utility and $d\alpha/d\theta$.[7]

[7] From (5), $S_\alpha(1, 0) = 0$. Therefore, if the first-order condition is sufficient, $\alpha(\theta)$ may be found by solving the differential equation (6), for $\theta \geq 0$, with initial condition $\alpha(0) = 1$.

We interpret Equation (6) by inspecting the marginal utility of incentives $v_\alpha(\alpha,\theta) = \mu(e^*(\alpha)) - \alpha\theta\sigma^2(f^*(\alpha,\theta))$. The risk aversion coefficient affects exclusively the second term, which represents the marginal cost of risk premium. The cross-derivative of indirect utility clarifies the relationship between marginal utility of incentives and risk aversion:

$$v_{\alpha\theta}(\alpha,\theta) = \underbrace{-\alpha\sigma^2(f^*(\alpha,\theta))}_{<0}\underbrace{-\alpha\theta\sigma^{2\prime}(f^*(\alpha,\theta))f^*_\theta(\alpha,\theta)}_{>0} \tag{7}$$

The first term is the direct effect and the second one is the variance reduction effect. The direct effect occurs as more risk-averse agents are more sensitive to the increase in variance when incentives increase; and the variance reduction effect reflects the higher risk reduction effort exerted by more risk-averse agents. As direct and variance reduction effects have opposite signs, $v_{\alpha\theta}(\alpha,\theta)$ may have any sign.[8] In the optimal contract, when direct effect dominates, agents with higher risk aversion have higher marginal cost of incentives and, for this reason, receive lower incentives to balance marginal cost and benefit; conversely, when variance reduction effect dominates, agents with higher risk aversion have lower marginal cost to the principal as they strongly reduce the variance and the principal provides more incentives. In the standard moral hazard models, in which the agent cannot control the risk of the project, only the direct effect exists, so the marginal cost of incentive associated with the risk premium increases with the agent's risk aversion and with the exogenous variance. Thus, the principal assigns lower-powered incentive contracts to more risk-averse agents or when risk is higher.

2.3 Risk and incentives

We now investigate the relationship between endogenous risk and risk aversion. As endogenous variance is given by $\sigma^2(f^*(\alpha(\theta),\theta))$, total differentiation gives,

$$\frac{d\sigma^2}{d\theta} = \sigma^{2\prime}(f^*(\alpha(\theta),\theta))f^*_\alpha(\alpha(\theta),\theta)\left[\frac{d\alpha}{d\theta} + \frac{f^*_\theta(\alpha(\theta),\theta)}{f^*_\alpha(\alpha(\theta),\theta)}\right] \tag{8}$$

Let $\hat{\alpha}(\theta; f)$ be the constant-risk curve associated with effort f, that is, $f^*(\hat{\alpha}(\theta; f), \theta) = f$. For a given f, the slope of $\hat{\alpha}(\theta)$ is

$$\frac{d\hat{\alpha}}{d\theta} = -\frac{f^*_\theta(\alpha(\theta),\theta)}{f^*_\alpha(\alpha(\theta),\theta)} < 0 \tag{9}$$

[8] Therefore, the single-crossing condition does not hold in this model. It will play an important role when we introduce adverse selection in the next section.

Therefore, Equation (8) states that $d\sigma^2/d\theta$ is determined by the interaction between contract curve and constant-risk curves; $d\sigma^2/d\theta$ is positive if $d\hat{\alpha}/d\theta > d\alpha/d\theta$ and negative if $d\hat{\alpha}/d\theta < d\alpha/d\theta$. As $d\hat{\alpha}/d\theta < 0$, the relationship between risk and risk aversion is positive only if incentives and risk aversion are negatively related. Moreover, as $f^*_\theta > 0$ and $f^*_\alpha > 0$, the reduction in incentives induced by the increase in risk aversion must be strong enough so that the effect of incentives dominates the effect of risk aversion on f.

The relationship between risk and incentives is the ratio of $d\alpha/d\theta$ and $d\sigma^2/d\theta$. Proposition 1 summarizes the relationship between endogenous risk, risk aversion and incentives.

Proposition 1 *Let $\alpha(\theta)$ be the incentive and $\sigma^2(f^*(\alpha(\theta),\theta))$ be the endogenous variance of the optimal contract. Then*

1. *The relationship between incentives and risk aversion has the same sign as $v_{\alpha\theta}(\alpha(\theta),\theta)$.*
2. *The relationship between endogenous risk and risk aversion has the same sign as*

$$\frac{v_{\alpha\theta}(\alpha(\theta),\theta)}{S_{\alpha\alpha}(\alpha(\theta),\theta)} - \frac{\alpha}{2\theta}$$

3. *The relationship between endogenous risk and incentives has the same sign as*

$$\left[\frac{v_{\alpha\theta}(\alpha(\theta),\theta)}{S_{\alpha\alpha}(\alpha(\theta),\theta)} - \frac{\alpha}{2\theta}\right] v_{\alpha\theta}(\alpha(\theta),\theta)$$

Three cases are possible. If $v_{\alpha\theta} > 0$, risk and incentives are negatively related as incentives increase and risk decreases with risk aversion. If $\alpha(2\theta)^{-1}S_{\alpha\alpha} < v_{\alpha\theta} < 0$, risk and incentives are positively related as incentives and risk decrease with risk aversion. And, if $v_{\alpha\theta} < \alpha(2\theta)^{-1}S_{\alpha\alpha}$, risk and incentives are negatively related as incentives decrease and risk increases with risk aversion.

Figure 1 provides a graphical interpretation of Proposition 1. Inspecting Equation (1), it is clear that f^* is determined only by $\alpha^2\theta$. As f^* is constant if and only if $\alpha^2\theta$ is constant, constant-risk curves are defined by $\hat{\alpha}(\theta) = \gamma/\sqrt{\theta}$, where γ is a constant.[9] The dotted lines in Figure 1 represent constant-risk curves. As $f^*(\alpha,\theta)$ is increasing in α and θ, risk is greater for low incentives and low risk

[9] We reach the same conclusion by solving the differential equation $d\hat{\alpha}/d\theta = -\alpha/2\theta$, derived from (9).

aversion. The thick line is the set of points where $v_{\alpha\theta}(\alpha, \theta) = 0$ and coincides with a constant-risk curve.[10] We assume $v_{\alpha\theta}(\alpha, \theta)$ is positive above and negative below the $v_{\alpha\theta}(\alpha, \theta) = 0$ curve.

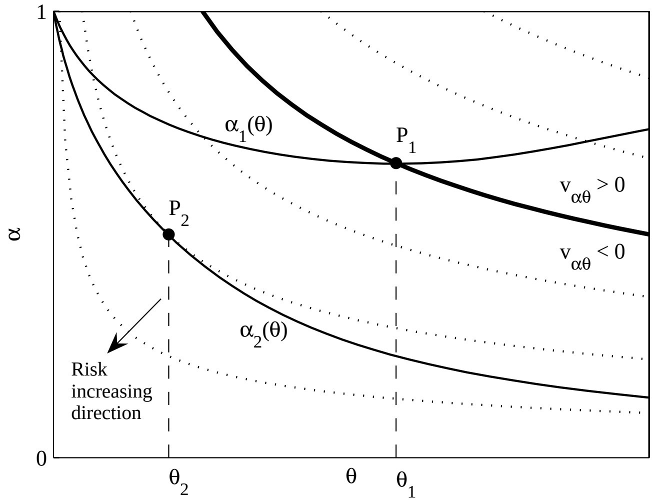

Figure 1
Optimal contracts and constant-risk curves

As an illustration, an optimal contract, $\alpha_1(\theta)$, is plotted. Let P_1 be the point where $v_{\alpha\theta}(\alpha_1, \theta) = 0$. As required by Equation (6), $\alpha_1(\theta)$ is increasing when $v_{\alpha\theta}(\alpha_1(\theta), \theta) > 0$ and decreasing when $v_{\alpha\theta}(\alpha_1(\theta), \theta) < 0$. For $\theta > \theta_1$, $v_{\alpha\theta}(\alpha_1(\theta), \theta) > 0$, therefore, for higher risk aversion, incentives are higher and the risk reduction effort increases. As a consequence, incentives and risk are negatively related. For $\theta < \theta_1$, the contract curve is flatter than the constant-risk curve. In this case, a moderate reduction in incentives is sufficient to compensate for an increase in risk aversion; as risk aversion dominates in the determination of the risk reduction effort, the risk decreases. Therefore, risk and incentives are positively related. The contract $\alpha_2(\theta)$ illustrates another case. At P_2, the contract curve is tangent to the constant-risk curve. For $\theta > \theta_2$, the reduction in incentives is strong enough to make the agent reduce effort in risk reduction and risk increases. In this case, risk and incentives are negatively related.

[10]The curve $v_{\alpha\theta}(\alpha, \theta) = 0$ coincides with a constant-risk curve as, according to Equations (3) and (7), $v_{\alpha\theta}(\alpha, \theta)/\alpha$ depends on α and θ only through $\alpha^2\theta$.

2.4 Exogenous risk

The analysis above is concerned with the endogenous risk, the risk that remains after the risk reduction effort. Assume now that there is an exogenous parameter of risk. Let σ_0^2 be the exogenous variance; this is the natural risk of the project, which would be observed if the agent exerted no effort to reduce risk. We rewrite the endogenous variance as a function of effort and of exogenous variance, $\sigma^2(f, \sigma_0^2)$, and assume $\sigma^2(0, \sigma_0^2) = \sigma_0^2$, $\partial\sigma^2/\partial f \leq 0$ and $\partial\sigma^2/\partial\sigma_0^2 > 0$. Analogously, for a given θ, we rewrite the indirect utility and social surplus as $v(\alpha, \sigma_0^2)$ and $S(\alpha, \sigma_0^2)$.

The following proposition relates exogenous risk and incentives.

Proposition 2 *For a given θ, the relationship between exogenous risk and incentives is negative if $\partial\sigma^2/\partial f = 0$, and has the same sign as*

$$-\frac{\partial\sigma^2}{\partial\sigma_0^2}\left[\frac{k''}{k'} - \left(\frac{\partial\sigma^2}{\partial f}\right)^{-1}\frac{\partial^2\sigma^2}{\partial f^2}\right] - \frac{\partial^2\sigma^2}{\partial f\partial\sigma_0^2} \tag{10}$$

if $\partial\sigma^2/\partial f < 0$.

The relationship between exogenous risk and incentives is a combination of the relationship between exogenous and endogenous risk and the relationship between endogenous risk and incentives. For a given θ, endogenous risk and incentives of optimal contracts are negatively related: if endogenous risk is higher, the marginal cost of incentives associated with the risk premium is higher and, consequently, the optimal level of incentives is lower.

The relationship between exogenous and endogenous risk is not necessarily positive, as the risk reduction effort is influenced by the exogenous risk. Expression (10) represents the interaction between two effects. The direct effect comes from $\partial\sigma^2/\partial\sigma_0^2 > 0$, that is the positive relationship between exogenous and endogenous risk, for a constant risk-reduction effort. The second effect is related to the risk reduction effort chosen by the agent. Depending on the sign of the cross-derivative $\partial^2\sigma^2/\partial f\partial\sigma_0^2$, the risk reduction effort may be positively or negatively related to the exogenous risk. If $\partial^2\sigma^2/\partial f\partial\sigma_0^2 > 0$, an increase in exogenous risk reduces the marginal benefit of effort, inducing the agent to decrease the effort in risk reduction. Therefore, in this case, endogenous risk increases with exogenous risk. Reversing the signs and following the same argument, we conclude that, if $\partial^2\sigma^2/\partial f\partial\sigma_0^2 < 0$, endogenous risk decreases with exogenous risk. The positive expression in brackets in (10) balances the relative strength of the two effects, taking into account the ratio between effort and risk at the agent's optimal choice of effort.

The following corollary is a straightforward consequence of Proposition 2 and provides a sufficient condition that depends exclusively on properties of the function $\sigma^2(f, \sigma_0^2)$.

Corollary 1 *If*

$$\frac{\partial \sigma^2}{\partial \sigma_0^2} \frac{\partial^2 \sigma^2}{\partial f^2} - \frac{\partial \sigma^2}{\partial f} \frac{\partial^2 \sigma^2}{\partial f \partial \sigma_0^2} \geq 0$$

then incentives and exogenous risk are negatively related.

The condition of Corollary 1 is satisfied by usual specifications of $\sigma^2\left(f, \sigma_0^2\right)$, such as $(\sigma_0 - f)^2$, $\sigma_0^2/(1+f)$ and $\sigma_0^2 \exp(-f)$. This result suggests that the negative relationship between risk and incentives predicted in Holmstrom and Milgrom (1987) is a quite robust property when risk is exogenous. However, as exogenous risk is hard to measure, empirical work may find positive or negative relation between risk and incentives.

2.5 The linear-quadratic specification

We specialize the model by assuming linear mean and quadratic cost.

Assumption 2 $\mu'(e) = m$ *and* $c''(e) = \bar{c}$, *where* m *and* $\bar{c}$ *are positive constants.*

Proposition 3 *Under Assumption 2, endogenous risk and incentives are positively related if and only if* $v_{\alpha\theta}(\alpha, \theta) < 0$ *and* $\alpha > \frac{1}{2}$.

To find numerical solutions of optimal contracts, we assume specific functions: the mean of profits is linear, and the variance and the cost functions are quadratic functions of efforts.

Assumption 3 $\mu(e) = me$, $\sigma^2(f) = (\sigma_0 - if)^2$, $c(e) = \frac{1}{2}e^2$ *and* $k(f) = \frac{1}{2}f^2$.

The binary variable i is either 0 or 1. When $i = 0$, the variance is not controllable and the model simplifies to the traditional Holmstrom-Milgrom single-task model; when $i = 1$, the variance is controllable.

2.5.1 Single-task

As a reference, we reproduce here the traditional result for uncontrollable risk. Under Assumption 3 with $i = 0$, effort choices $e^* = m\alpha$ and $f^* = 0$ follow from (1). Using (5) and $S_\alpha(\alpha, \theta) = 0$, the optimal incentive is

$$\alpha = \frac{m^2}{m^2 + \theta\sigma_0^2}$$

This result is in accordance with Proposition 2. As risk is not controllable, the endogenous risk coincides with exogenous risk and, for a given θ, the relationship between risk and incentives is negative. Note that as $\sigma^{2\prime}(f) = 0$ does not satisfy Assumption 1, Proposition 1 and 3 are not applicable to this case.

2.5.2 Controllable risk

To study the controllable risk case, we restrict the values of risk aversion to a range in which the solutions of the optimization problems are characterized by the first-order condition.

Assumption 4 *The parameter of risk aversion is restricted to* $0 < \theta < 4$.

The following proposition summarizes the properties of the optimal contract.

Proposition 4 *Under Assumptions 3 and 4 with* $i = 1$*, the optimal contract exhibits the following properties:*

1. *The incentive assigned to the agent with risk aversion* θ *is uniquely defined by*

$$m^2(1-\alpha)(1+\alpha^2\theta)^2 - \alpha\theta\sigma_0^2 = 0$$

2. *Exogenous risk and incentives are negatively related.*

3. *Endogenous risk and incentives are positively related if and only if* $1/2 < \alpha < 1/\sqrt{\theta}$.

4. *Endogenous risk and incentives are positively related if and only if*

$$\frac{4(\sqrt{\theta}-1)}{\theta} < \frac{\sigma_0^2}{m^2} < \frac{1}{\theta}\left(1+\frac{\theta}{4}\right)^2$$

Figure 2 shows the relevant regions, together with five instances of optimal contracts. Let $s^2 = \sigma_0^2/m^2$. In Region A, $v_{\alpha\theta} > 0$, incentive is increasing and variance is decreasing in θ. In Region B, $v_{\alpha\theta} < 0$, both incentive and variance are decreasing in θ. And, in Region C, $v_{\alpha\theta} < 0$, incentive is decreasing and variance is increasing in θ. Therefore, risk and incentives are positively correlated in Region B and negatively correlated in Regions A and C. Risk and incentives are positively correlated when risk aversion is low and associated with high incentives.

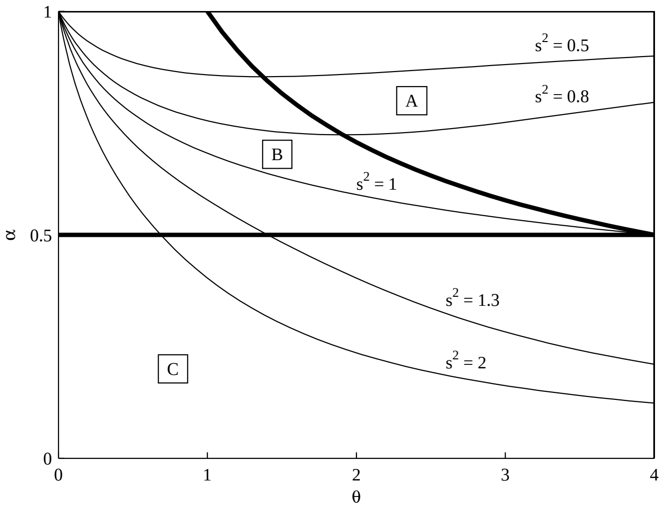

Figure 2
Optimal contracts. $s^2 = \sigma_0^2/m^2$

Figure 3 shows the combinations of parameters that result in increasing or decreasing relationship between risk and incentives, according to Proposition 4.4. For agents with high risk aversion, positive correlation is possible if σ_0^2/m^2 has intermediate values, but this is a less frequent situation, the higher the risk aversion. As $\sigma^{2\prime}(f)$ increases with σ_0^2, it is easier for the agent to reduce risk, the lower the exogenous variance. For intermediate values of exogenous risk aversion, incentives and endogenous risk are decreasing with respect to risk aversion. If the exogenous risk is low, as in Region A, the principal gives more incentives to high risk aversion agents as they are able to reduce the risk premium of the incentive. If the exogenous risk is high, as in Region C, endogenous risk increases with risk aversion because incentives decrease and the agent exerts less effort in risk reduction.

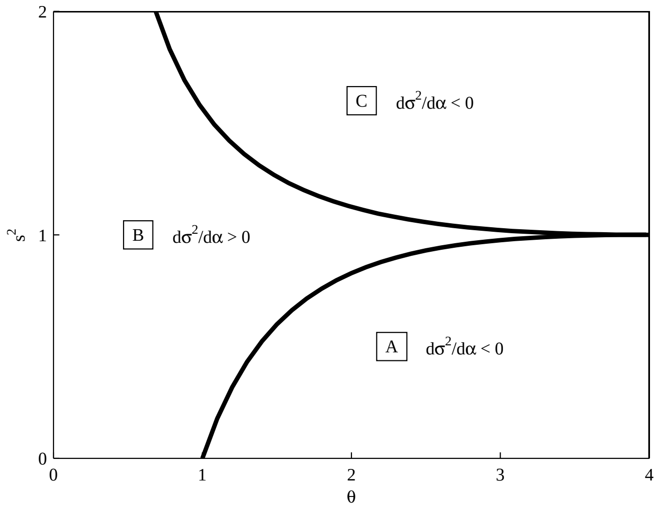

Figure 3
Parameters and the risk-incentive trade-off. $s^2 = \sigma_0^2/m^2$

3. Adverse Selection and Moral Hazard

In this section we examine the optimal contract and the relationship between incentives and risk, when the risk aversion is private information of the agent. The characterization of the optimal contract is more complex and we do not provide a general analysis as the one developed for the observable risk aversion case. Instead, we present basic properties of the model and illustrative examples that show that positive and negative relationship between incentives and risk may be observed when there is adverse selection before moral hazard.

The model is similar to the one in Section 2, but it assumes that risk aversion is the agent's private information. The principal knows that θ is uniformly distributed on $\Theta = [\theta_a, \theta_b]$, and she designs a menu of linear contracts taking into account the participation and incentive compatibility constraints.[11] The timing of the problem is: (1) the agent learns his type, (2) the principal offers a menu of contracts $\{\alpha(\theta), \beta(\theta)\}_{\theta\in\Theta}$, (3) the agent chooses a contract, and (4) exerts effort accordingly, (5) the firm produces profit z, (6) the agent receives $w = \alpha z + \beta$ and the principal earns the net profit, $z - w$.

We can now divide the problem into two parts. First, for a given contract (α, β), the agent chooses the level of effort in mean increase and in risk reduction.

[11]See Sung (2005) for a discussion on optimality of linear contracts in this setting.

This part is exactly the same as the agent's problem in the pure moral hazard problem. The optimal effort choices are characterized by (1) and the indirect utility of the agent is $V(\alpha, \beta, \theta) = \beta + v(\alpha, \theta)$, where $v(\alpha, \theta)$ is defined by (4). Second, the principal solves the adverse selection problem, by designing a menu of contracts, $(\alpha(\theta), \beta(\theta))$, that maximizes her expected utility subject to incentive compatibility and participation constraints,

$$V(\alpha(\theta), \beta(\theta), \theta) \geq V(\alpha(\hat{\theta}), \beta(\hat{\theta}), \theta), \text{ for all } \theta, \hat{\theta} \in \Theta, \text{ and} \quad (11)$$

$$V(\alpha(\theta), \beta(\theta), \theta) \geq 0 \text{ for all } \theta \in \Theta \quad (12)$$

The incentive schedule $\alpha(\theta)$ is implementable if there is a function $\beta(\theta)$ that satisfies (11) and (12). Guesnerie and Laffont (1984) characterize the optimal contract under the single-crossing condition, that is, the cross-derivative $v_{\alpha\theta}$ has constant sign. In this case, the monotonicity condition,

$$v_{\alpha\theta}(\alpha(\theta), \theta)\alpha'(\theta) \geq 0 \quad (13)$$

is a sufficient condition for implementability. In our model, the single-crossing condition does not hold and the monotonicity condition is necessary but not sufficient. The complete characterization of the optimal contract without the single-crossing property is out of the scope of this paper. A detailed analysis is found in Araujo and Moreira (2001b).

3.1 Relaxed solution

A simple case occurs when the optimal contract is the solution to the relaxed problem. The relaxed problem is the unconstrained maximization of the virtual surplus, $R(\alpha, \theta) = S(\alpha, \theta) + (\theta - \theta_a)v_\theta(\alpha, \theta)$. The solution $\alpha_r(\theta) = \arg\max_\alpha R(\alpha, \theta)$ is called relaxed solution. If $\alpha_r(\theta)$ is an implementable contract, then it is the optimal contract.

3.2 The linear quadratic specification

We proceed by assuming the linear-quadratic specification in Assumption 3 and presenting some representative examples. Except for the last example, the optimal contracts coincide with the relaxed solutions.

3.2.1 Single-task

First, we analyze the specification in Assumption 3 with $i = 0$. The agent controls the mean of the profits but cannot control their variance. The objective is to certify that the possibility of variance reduction is a necessary element of the model to generate a positive relationship between incentives and risk.

The first-order conditions of the agent's problem provide the optimal efforts, $e^* = m\alpha$ and $f^* = 0$. The single-crossing property holds for this case, since $v_{\alpha\theta} = -\alpha\sigma_0^2 < 0$. An agent with low risk aversion has high marginal utility of incentive and may choose a high-powered incentive contract. The relaxed solution is

$$\alpha_r(\theta) = \frac{m^2}{m^2 + (2\theta - \theta_a)\sigma_0^2}$$

As $\alpha_r(\theta)$ is decreasing, the monotonicity condition (13) holds and consequently $\alpha_r(\theta)$ is the optimal menu of contracts. The relationship between α and σ_0^2 is negative, given θ. Therefore, adverse selection before moral hazard is not sufficient to change the traditional trade-off between risk and incentives. More risk increases the principal's marginal cost because she has to pay the risk premium and the informational rent to the agent.

3.2.2 Controllable risk

In the controllable risk case, Assumption 3 with $i = 1$, the variance and the mean are controlled by the agent. The single-crossing property does not hold. For high risk-aversion agents, the variance reduction effect dominates and $v_{\alpha\theta} > 0$. An agent with a higher degree of risk aversion has a higher marginal utility of incentive and chooses contracts with higher incentives.

Proposition 5 *Under Assumption 3, exogenous risk and incentives are negatively related in the relaxed solution.*

The result in Proposition 5 is restricted to the relaxed solution of the linear-quadratic case, but suggests that the negative relationship between risk and incentives found in Proposition 2 is preserved when adverse selection is added to the model.

The relaxed problem is solved for two representative cases in which the relaxed solution is the optimal contract and generates positive and negative relationships between incentives and risk. The parameter values $\sigma_0 = 0.91$ and $m = 1$ are the same for both cases, and the interval $\Theta = [\theta_a, \theta_b]$ changes for each case.

Figure 4 presents the relevant curves for $\Theta = [2.5, 3.5]$. The left graph shows the optimal menu of contracts, $\alpha^*(\theta)$. The cross-derivative $v_{\alpha\theta}$ is positive to the right of the dotted line $\alpha_0(\theta)$. The relaxed solution $\alpha_r(\theta)$ is increasing in Θ and is implementable. The right graph is the corresponding plot for endogenous risk and incentives. An agent with higher risk aversion exerts more effort in risk reduction and this behavior reduces the marginal cost of risk premium. This effect more than compensates for the increase in marginal cost due to higher risk aversion. The net effect is that more risk-averse agents choose higher-powered incentive contracts and the relationship between risk and incentives is negative.

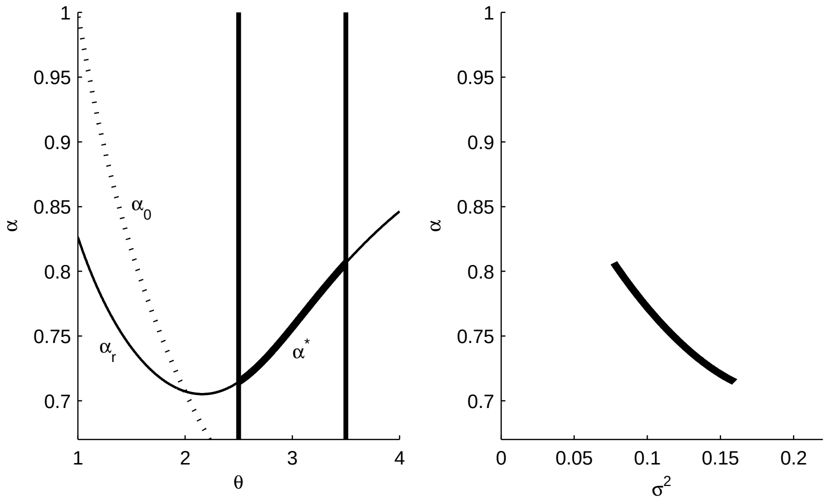

Figure 4
Optimal contract. $\Theta = [2.5, 3.5]$

Figure 5 shows the same curves for a set of types with lower risk aversion, $\Theta = [0.5, 1.4]$. The relaxed solution is implementable and coincides with the optimal contract, but in this case $v_{\alpha\theta}$ is negative and the relationship is reversed. More risk-averse agents have higher marginal cost of incentives, thus they prefer lower-powered incentive contracts. At the same time, more risk-averse agents exert more effort in risk reduction and the variance is lower. Thus, risk and incentives are positively related.

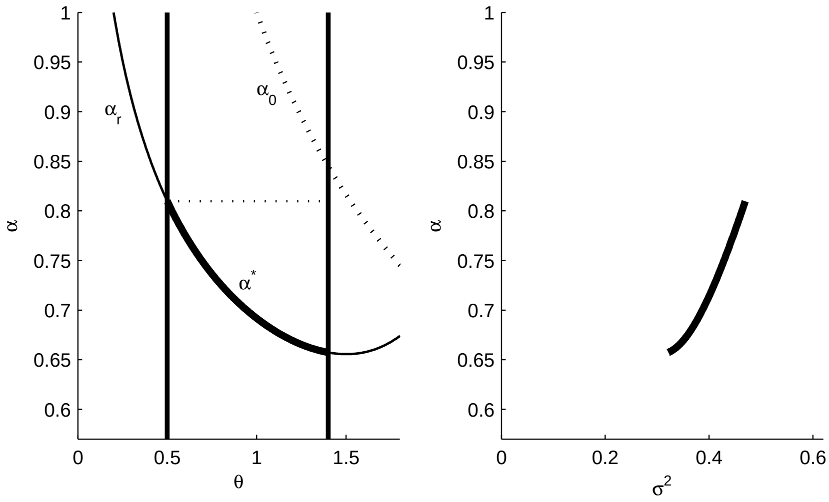

Figure 5
Optimal contract. $\Theta = [0.5, 1.4]$

3.2.3 The non-relaxed solution without the single-crossing property

For a broader interval of types that encompasses positive and negative values of $v_{\alpha\theta}$, the relaxed solution may not be implementable. As examined in Araujo and Moreira (2001b), the optimal contract without the single-crossing property may present a U-shaped form, therefore a discrete set of agent's types may choose the same contract. This situation is called discrete pooling. The optimal contract is the best implementable combination of the relaxed solution, discrete pooling and bunching intervals. In Figure 6, we present the optimal contract for $\sigma_0 = 0.91$, $m = 1$, and $\Theta = [0.7, 3.0]$.[12] Incentives and risk aversion are positively related for more risk-averse agents and negatively related for less risk-averse agents. The U-shape of the optimal contract is also present in the risk incentive graph. The sensitivity of incentives to variations in the exogenous risk was numerically calculated and negative values were found for $d\alpha/d\sigma_0$. This result suggests that the negative relationship between exogenous risk and incentives is a general property of the model for usual specifications of functions.

[12]Computer code is provided upon request.

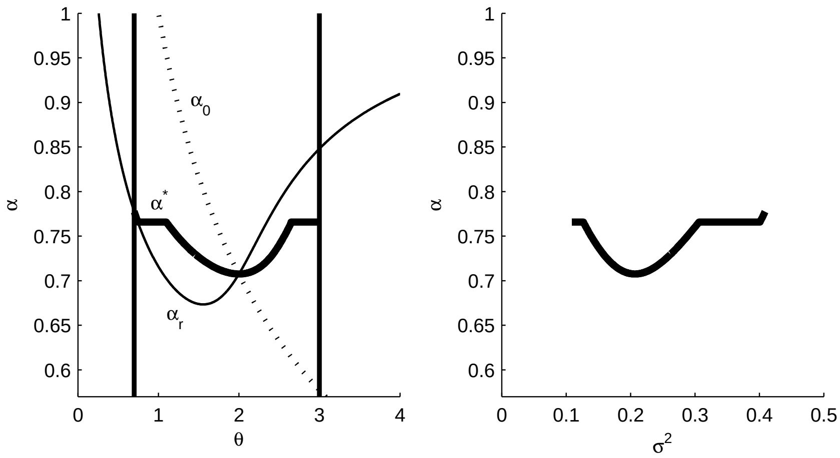

Figure 6
Optimal contract. $\Theta = [0.7, 3.0]$

4. Conclusion

The negative relationship between risk and incentives, found in standard models of moral hazard, may be reversed if we allow the agent to control the variance. In the model with moral hazard in which risk aversion is observable and where the agent may exert costly efforts to increase the mean as well as to reduce variance, we find that effort in variance reduction is an increasing function of both incentives and risk aversion.

The marginal utility of incentives may be decreasing or increasing in risk aversion. As incentives increase wage variance, the marginal utility of incentives tends to be lower for higher risk aversion because the marginal cost from the risk premium is higher. However, as the effort in variance reduction increases with incentives, when risk aversion is sufficiently high, the marginal cost from the risk premium may decrease and the marginal utility of incentives may increase with the risk aversion.

In the moral hazard model, when the marginal utility of incentives is decreasing in risk aversion, the principal maximizes the social surplus by giving lower incentives to a more risk-averse agent. Conversely, for increasing marginal utility, incentives and risk aversion are positively related. In the model with adverse selection before moral hazard, the incentive compatibility constraint leads to the same relationship between the marginal utility and incentives found in the pure moral hazard model.

The relationship between risk and incentives is determined by the interaction of the effects described in the previous paragraphs. Three cases may be identified. First, when marginal utility of incentives is increasing in risk aversion, incentives and risk aversion are positively related. And as effort in variance reduction is increasing in both incentives and risk aversion, the relationship between incentives and risk is negative. The other two cases occur when the marginal utility of incentives is decreasing in risk aversion. As incentives and risk aversion are negatively related, the effect in the variance reduction effort is ambiguous. If the incentive effect dominates, the variance reduction effort increases with incentives and the relationship between incentives and risk is negative. If the risk aversion effect dominates, the variance reduction effort decreases with incentives and the relationship between incentives and risk is positive. We found that the positive relationship between incentives and risk is more likely when risk aversion is low and incentives are high.

The analysis above refers to the endogenous risk, which is the empirically relevant case, since endogenous risk is observable. The relationship between the exogenous risk and incentives remains negative for the pure moral hazard model and the numerical calculations for the linear-quadratic specification suggest the same result for the model with adverse selection before moral hazard.

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Appendix A

Proof of the Propositions

Proof of Proposition 1 As $\alpha(\theta)$ maximizes the social surplus, $S_{\alpha\alpha}(\alpha(\theta),\theta) < 0$, part (1) is a consequence of Equation (6) and part (2) results from Equations (2), (3), (6) and (8). Part (3) is the combination of parts (1) and (2). □

Proof of Proposition 2 As the optimal contract is characterized by $S_\alpha(\alpha,\sigma_0^2) = 0$, the relationship between exogenous risk and incentives is given by

$$\frac{d\alpha}{d\sigma_0^2} = -\frac{v_{\alpha\sigma_0^2}(\alpha,\sigma_0^2)}{S_{\alpha\alpha}(\alpha,\sigma_0^2)}$$

which has the same sign as $v_{\alpha\sigma_0^2}$.

According to the envelope theorem $v_\alpha(\alpha,\sigma_0^2) = \mu(e^*(\alpha)) - \alpha\theta\sigma^2(f^*(\alpha,\sigma_0^2),\sigma_0^2)$. Then,

$$v_{\alpha\sigma_0^2}(\alpha,\sigma_0^2) = -\alpha\theta\left[\frac{\partial\sigma^2}{\partial f}f^*_{\sigma_0^2} + \frac{\partial\sigma^2}{\partial\sigma_0^2}\right]$$

which is negative if $\partial\sigma^2/\partial f = 0$. From Equation (1),

$$f^*_{\sigma_0^2} = -\frac{1}{2}\alpha^2\theta\frac{\partial^2\sigma^2}{\partial f\partial\sigma_0^2}\left[k'' + \frac{1}{2}\alpha^2\theta\frac{\partial^2\sigma^2}{\partial f^2}\right]^{-1}$$

We also use (1) to eliminate $\alpha^2\theta$, and after algebraic manipulations,

$$v_{\alpha\sigma_0^2}(\alpha,\sigma_0^2) = \alpha\theta\left[-\frac{\partial\sigma^2}{\partial\sigma_0^2} - \frac{\frac{\partial^2\sigma^2}{\partial f\partial\sigma_0^2}}{\frac{k''}{k'} - \left(\frac{\partial\sigma^2}{\partial f}\right)^{-1}\frac{\partial^2\sigma^2}{\partial f^2}}\right]$$

Proof of Proposition 3 By Proposition 1, risk and incentives are positively related if and only if $v_{\alpha\theta} < 0$ and $v_{\alpha\theta} > \alpha(2\theta)^{-1}S_{\alpha\alpha}$. Thus, we have to show the second condition. By a straightforward application of Assumption 2, $e^*_\alpha = m/\bar{c}$. Then, by direct differentiation,

$$S_\alpha(\alpha,\theta) = (1-\alpha)m^2/\bar{c} - \alpha\theta\sigma^2(f^*(\alpha,\theta))$$

$$S_{\alpha\alpha}(\alpha,\theta) = -m^2/\bar{c} - \theta\sigma^2(f^*(\alpha,\theta)) - \alpha\theta\sigma^{2\prime}(f^*(\alpha,\theta))f^*_\alpha(\alpha,\theta)$$

and

$$v_{\alpha\theta}(\alpha,\theta) = S_{\alpha\theta}(\alpha,\theta) = -\alpha\sigma^2(f^*(\alpha,\theta)) - \alpha\theta\sigma^{2\prime}(f^*(\alpha,\theta))f^*_\theta(\alpha,\theta)$$

Substituting in the second condition and rearranging,

$$\theta\sigma^2(f^*(\alpha,\theta)) + \theta\sigma^{2\prime}(f^*(\alpha,\theta))[2\theta f^*_\theta(\alpha,\theta) - \alpha f^*_\alpha(\alpha,\theta)] < m^2/\bar{c}$$

As $S_\alpha(\alpha,\theta) = 0$, the first term on the left hand side is $m^2(1-\alpha)/(\bar{c}\alpha)$, and by Equations (2) and (3), the second term is zero. Therefore, $\alpha > \frac{1}{2}$. □

Proof of Proposition 4 Using the functions specified in Assumption 3, the cross-derivative of the indirect utility is

$$v_{\alpha\theta}(\alpha,\theta) = -\frac{\alpha(1-\alpha^2\theta)}{(1+\alpha^2\theta)^3}\sigma_0^2$$

and the social surplus becomes

$$S(\alpha,\theta) = \left(1-\frac{\alpha}{2}\right)\alpha m^2 - \frac{\alpha^2\theta}{2(1+\alpha^2\theta)}\sigma_0^2$$

Differentiating with respect to α,

$$S_\alpha(\alpha,\theta) = m^2(1-\alpha) - \frac{\alpha\theta\sigma_0^2}{(1+\alpha^2\theta)^2}$$

and the equation in property (1) results from the first-order condition of the principal's problem, $S_\alpha(\alpha,\theta) = 0$.

Note that $S_\alpha(\alpha,\theta)$ is positive when $\alpha \leq 0$ and negative when $\alpha \geq 1$. Thus, by continuity, for each θ, the social surplus has a maximum in the interval $(0,1)$, which satisfies $S_\alpha(\alpha,\theta) = 0$ and $S_{\alpha\alpha}(\alpha,\theta) < 0$. The uniqueness may be established by analyzing the properties of the second-order condition. By continuity, if there are two distinct solutions α_1 and α_2 for first- and second-order conditions, then there must be a local minimum α_3 such that $S_\alpha(\alpha_3,\theta) = 0$ and $S_{\alpha\alpha}(\alpha_3,\theta) > 0$. By differentiation,

$$S_{\alpha\alpha}(\alpha,\theta) = -m^2 - \frac{(1-3\alpha^2\theta)\theta\sigma_0^2}{(1+\alpha^2\theta)^3}$$

And using the first-order condition to eliminate σ_0^2,

$$S_{\alpha\alpha}(\alpha(\theta),\theta) = \frac{[\alpha^2\theta(3-4\alpha)-1]m^2}{\alpha(1+\alpha^2\theta)}$$

For $\theta < 4$, $S_{\alpha\alpha}(\alpha(\theta),\theta) < 0$. Thus, under Assumption 4, $S_\alpha(\alpha,\theta) = 0$ never defines a local minimum of the social surplus.

Properties (2) and (3) follow from Propositions 2 and 3. Let $q(\alpha,\theta) = (1-\alpha)(1+\alpha^2\theta)^2/(\alpha\theta)$. From the first-order condition, $\sigma_0^2/m^2 = q(\alpha,\theta)$. By property (2), $q_\alpha(\alpha,\theta) < 0$. As $q(1/2,\theta) = (1+\theta/4)^2/\theta$ and $q(1/\sqrt{\theta},\theta) = 4(\sqrt{\theta}-1)/\theta$, property (3) implies property (4). □

Proof of Proposition 5 Under Assumption 3, the virtual surplus of the problem is

$$R(\alpha,\theta) = \frac{\alpha(2-\alpha)}{2}m^2 - \frac{\alpha^2(\alpha^2\theta^2+2\theta-\theta_a)}{2(1+\alpha^2\theta)^2}\sigma_0^2$$

The derivative with respect to α is

$$R_\alpha(\alpha,\theta) = (1-\alpha)m^2 - \frac{\alpha[\theta(1+\alpha^2\theta_a)+(\theta-\theta_a)]}{(1+\alpha^2\theta)^3}\sigma_0^2$$

and, for the relaxed solution $\alpha_r(\theta)$, $R_\alpha(\alpha_r(\theta),\theta) = 0$ and $R_{\alpha\alpha}(\alpha_r(\theta),\theta) < 0$. Note that $R_\alpha(\alpha,\theta) > 0$, for $\alpha < 0$, and $R_\alpha(\alpha,\theta) < 0$, for $\alpha > 1$. So the relaxed problem has a solution in the interval $(0,1)$. If $R(\cdot,\theta)$ is not concave in α, the incentive that maximizes the virtual surplus must be correctly chosen among the solutions of the first-order condition.

By fixing a value of θ, we can write R_α as a function of α and σ_0^2. By differentiation, $R_{\alpha\sigma_0}(\alpha,\sigma_0) < 0$, and, as $R_{\alpha\alpha}(\alpha_r(\sigma_0^2),\sigma_0^2) < 0$, the application of the implicit function theorem to $R_\alpha(\alpha_r(\sigma_0^2),\sigma_0^2) = 0$ gives $d\alpha_r/d\sigma_0 < 0$. That is, for a given θ, an increase in the exogenous risk reduces incentives in the relaxed solution. □

Appendix B

Optimal Contract Without the Single-Crossing Property

The adverse selection problem is to find the functions $\alpha(\cdot)$ and $\beta(\cdot)$ that maximize

$$E[(1-\alpha(\theta))\mu(e^*(\alpha(\theta))) - \beta(\theta)] \tag{14}$$

subject to (11) and (12). The expectation is taken with respect to the distribution of θ. The principal's maximization problem for the pure moral hazard case studied in Section 2 is equivalent to this problem without the incentive compatibility constraint (11).

Assuming that $\alpha(\cdot)$ and $\beta(\cdot)$ are differentiable, the incentive compatibility constraint implies the first-order condition

$$v_\alpha(\alpha(\theta),\theta)\alpha'(\theta) + \beta'(\theta) = 0 \tag{15}$$

The second-order condition gives $v_{\alpha\alpha}(\alpha(\theta),\theta)[\alpha'(\theta)]^2 + v_\alpha(\alpha(\theta),\theta)\alpha''(\theta) + \beta''(\theta) \leq 0$, and, after differentiating (15) with respect to θ, it simplifies to the condition (13), which implies the monotonicity of $\alpha(\theta)$, in the single-crossing context. Given the menu of implementable contracts $\{\alpha(\theta),\beta(\theta)\}_{\theta\in\Theta}$, the level of utility achieved by the agent with risk aversion θ is the informational rent $r(\theta) = v(\alpha(\theta),\theta) + \beta(\theta)$. Using the first-order condition, we get $r'(\theta) = v_\theta(\alpha(\theta),\theta)$. As $v_\theta(\alpha,\theta) = -\frac{1}{2}\alpha^2\sigma^2(e^*) < 0$, the participation constraint is binding for the agent with the highest level of risk aversion, that is, $r(\theta_b) = 0$. Thus, the fixed component of the wage can be isolated by the integration of $r'(\theta)$,

$$\beta(\theta) = -\int_\theta^{\theta_b} v_\theta(\alpha(\tilde{\theta}),\tilde{\theta})d\tilde{\theta} - v(\alpha(\theta),\theta) \tag{16}$$

and we can eliminate $\beta(\theta)$ from (14). As types are uniformly distributed, applying Fubini's theorem,

$$E\left[\int_\theta^{\theta_b} v_\theta(\alpha(\tilde{\theta}),\tilde{\theta})d\tilde{\theta}\right] = E\left[v_\theta(\alpha(\theta),\theta)(\theta-\theta_a)\right]$$

and the principal's objective function can be rewritten as $E[R(\alpha,\theta)]$.

As $V(\alpha,\beta,\theta) = v(\alpha,\theta)+\beta$ and using (16), the incentive compatibility condition may be written in terms of $v_{\alpha\theta}$,

$$\begin{aligned}
&V(\alpha(\theta),\beta(\theta),\theta) - V(\alpha(\hat{\theta}),\beta(\hat{\theta}),\theta) \\
&= -\int_{\theta}^{\theta_b} v_\theta(\alpha(\tilde{\theta}),\tilde{\theta})d\tilde{\theta} + \int_{\hat{\theta}}^{\theta_b} v_\theta(\alpha(\tilde{\theta}),\tilde{\theta})d\tilde{\theta} + v(\alpha(\hat{\theta}),\hat{\theta}) - v(\alpha(\hat{\theta}),\theta) \\
&= -\int_{\theta}^{\hat{\theta}} v_\theta(\alpha(\tilde{\theta}),\tilde{\theta})d\tilde{\theta} + \int_{\theta}^{\hat{\theta}} v_\theta(\alpha(\hat{\theta}),\tilde{\theta})d\tilde{\theta} = \int_{\theta}^{\hat{\theta}}\int_{\alpha(\tilde{\theta})}^{\alpha(\hat{\theta})} v_{\alpha\theta}(\tilde{\alpha},\tilde{\theta})\, d\tilde{\alpha}\, d\tilde{\theta} \geq 0
\end{aligned}$$

The contract $\alpha(\theta)$ is implementable if $\alpha(\theta)$ is non-decreasing and $v_{\alpha\theta} > 0$ in the region of integration or if $\alpha(\theta)$ is non-increasing and $v_{\alpha\theta} < 0$ in the region of integration. This condition is satisfied by the examples in Figures 4 and 5.