# Immunization of Fixed-Income Portfolios Using an Exponential Parametric Model[*]

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**Abstract**

Litterman and Scheinkman (1991) showed that even a *duration* immunized fixed-income portfolio (neutral) can bear great losses and, therefore, propose hedging the portfolio using a principal component's analysis. The problem is that this approach is only possible when the interest rates are observable. Therefore, when the interest rates are not observable, as is the case of most international and domestic debt markets in many emerging market economies, it is not possible to apply this method directly. The present study proposes an alternative approach: hedging based on factors of a parametric term structure model. The immunization done using this approach is not only simple and efficient but also equivalent to the immunization procedure proposed by Litterman and Scheinkman when the rates are observable. Examples of the method for hedging and leveraging operations in the Brazilian inflation-indexed public debt securities market are presented. This study also describes how to construct and to price portfolios that replicate the model risk factors, which makes possible to extract some information on the expectations of agents in regards to future behavior of the interest rate curve.

*Keywords:* Fixed Income, Term Structure, Asset Pricing, Inflation Expectation, Immunization, Hedge, Emerging Markets, Debt, Derivatives.

*JEL Codes:* E43, G12, G13.

## 1. Introduction

The interest rate term structure provides a connection between interest rates and maturities. Its uses are manifold: pricing of fixed-income instruments and interest rate derivatives, portfolio management, risk management, rates forecasting, economic growth forecasting, estimation of risk premium, and monetary policy actions, among others. Due to its many uses, its estimation and the understanding of its movement in time is of considerable importance for practitioners, academics and central bankers. In the words of Alan Greenspan, former president of the Federal Reserve: *"the broadly unanticipated behavior of world bond markets remains a conundrum"*.[1]

The fact that the movements of the interest rate curve affect the price of fixed-income assets, together with the uncertainty surrounding the future interest rate curve, makes it necessary for fixed income managers to protect themselves against unexpected movements in the yield curve. To this end, several immunization techniques have been proposed. Fisher and Weil (1971) showed that, under the hypothesis of parallel changes in the interest rate curve, immunization is reached when the horizon of the investment is equal to the *duration* of the portfolio. Bierwag (1977) shows that duration-based hedging may not produce an efficient immunization when different hypothesis regarding the stochastic process of the interest rate are observed (multiplicative shocks and discrete compositions of the rates) and proposes an immunization based on an adjusted *duration*. Khang (1979) proposes a *duration* measure under the hypothesis that the short term interest rates vary more than the long term ones. The problem of these approaches lies in the fact that the accuracy of their strategy depends on a specific behavior of the term structure.

An alternative approach that is a generalization of the previous ones was proposed by Litterman and Scheinkman (1991) and consists in using principal component analysis.[2] They show that three principal components are enough to describe nearly all the historical movements of the American interest rate curve.[3] These three factors were denominated by them as: level, slope and curvature (or convexity). Additionally, they showed that portfolios immunized using *duration* (technique still widely used in the Brazilian market) eliminate the risk of the level factor, but not of the slope and curvature factors. As an example they built a portfolio composed of three securities that, although neutral in *duration*, generated a

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[1]Speech given at the Unite States Congress in February 2005. At that time, Alan Greenspan was the president of the Federal Reserve.

[2]The latent orthogonal factors which explain the totality of the historical variation in the time series built from a linear combination of the interest rate series of different maturities.

[3]Some examples of studies that demonstrate this result to other countries are: Barber and Cooper (1996) for the American spot curve, Bühler and Zimmermann (1996) for the Swiss and German spot curves, D'Ecclesia and Zenios (1994) for Italy, Golub and Tilman (1997) for the American curve, Kärki and Reyes (1994) for Germany, Switzerland and USA, and Lekkos (2000) for the German, England and Japan forward curves.

considerable loss in a period of one month. That position was especially sensitive to the curvature factor. Finally, they used the proposed hedging technique through principal components for the same portfolio, considerably improving the result of the operation. That is, they demonstrated the importance of considering different movements to obtain an effective immunization. The problem in that approach is that it needs the interest rates to be directly observables, which does not happen in all fixed-income markets. When the interest rates are not directly observable, an intermediate step – of building the interest rate curve – is necessary.[4]

To work around this problem, this study proposes an exponential parametric model for interest rates that simultaneously builds the movements of the curve (a rotation of the principal components) and allows undertaking the immunization of a fixed-income portfolio based on the constructed multiple movements. We theoretically derivethe necessary conditions to carry out a hedge or to trade the movements of the curve (level, slope and curvature) using the proposed parametric model. We additionaly show, through scenario analysis, the empirical applicability of the derived hedge conditions using the Brazilian public debt bonds indexed to CPI inflation. The parametric model is an extension of the Diebold and Li (2006) model proposed by Almeida et al. (2007), that presents desirable forecasting characteristics and that allows us to capture the high volatility of the interest rates of an emergent market economy such as Brazil.

Diebold and Li (2006) use variations of the exponential components of the framework proposed by Nelson and Siegel (1987) to model the spot rates, instead of the forward rates, additionally, they use an autoregressive model (AR(1)) for each of the factors to carry out a forecasting exercise of the American term structure. The model is interesting because of the interpretation of the factors, similarly to Litterman and Scheinkman, of level, slope and convexity. Additionally, it conforms to the stylized factors of the interest rate curve that were recorded throughout time and produces one year ahead forecastings that are notably better, for all maturities, in absolute or relative terms, than the standard reference models.[5] Almeida et al. (2007), compare the Diebold and Li model to an extended model (in which the fourth factor, representing a second curvature, is added) to analyze the importance of the curvature movement in the interest rates curve's forecasting. They report that the new factor increases the ability to generate more volatile and non-linear interest rates, and, for this reason, in an experiment using data from Brazilian term structure (DI Futuro contracts), they achieve better interest rates forecastings than the Diebold-Li model, especially for shorter maturities (1 day, 1

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[4]Some statistical models used for the construction of the interest rate curves are found in McCulloch (1971), Langetieg and Smoot (1981), Vasicek and Fong (1982), Nelson and Siegel (1987, 1988), Svensson (1995), Almeida et al. (1998).

[5]Some examples are: random walk, slope regression, Fama-Bliss forward-rate regression, Cochrane-Piazzesi forward curve regression, AR(1) on the rates, VAR(1) on the rates, VAR(1) on the rate variations, error correction model (ECM) with a common tendency, ECM with two common tendencies and, finally, AR(1) with tree statistical principal components.

month and 3 months). Additionally, they show that by adding a second curvature to the Diebold and Li model the risk premium structures changes, producing better forecastings of the bond excess returns.

Some of the studies related to the present one are the immunization models using parametric duration and the immunization model using $M$-vector. The firsts assume that the term structure can be represented by a function of few risk factors multiplied by their loadings – usually polynomials and exponential functions – and base the hedging strategy in a first order multivariate Taylor expansion of the fixed income portfolio value. Some examples of this approach using polynomials are: Cooper (1977), Garbade (1985), Chambers et al. (1988) and Prisman and Shores (1988). Willner (1996) carries out an immunization using the model described in Nelson and Siegel (1987). Bravo and Silva (2005) use the model described in Svensson (1995) and derive the second order conditions under which the portfolio convexity has a positive influence in the construction of the hedge.[6] The fundamental difference between the parametric *duration* models and the one proposed in the present study is that the first does not implement an immunization when the interest rate curve is unobservable, which is the aim of our proposal.

Another procedure related to our model is the immunization using $M$-vector proposed by Nawalka and Chambers (1997), a generalization of the $M$-square concept proposed by Vasicek and Fong (1984). The model is based on a Taylor expansion of the return function of a portfolio evaluated in a given investment horizon, generating an interest rate dispersion measure, which, when minimized, immunizes the fixed-income portfolio against parallel and non-parallel changes in the interest rates. Bravo and Silva (2006), in a recent empirical study, construct their curve using the Nelson and Siegel parametric model and investigate the efficiency of several immunization strategies for the Portuguese public debt market, such as: $M$-vector of several dimensions, *duration*, *maturity*,[7] *maturity-bullet* and *maturity-barbell*.[8] They conclude that multifactor or single factor immunization strategies eliminate the largest part of the interest rate risk subjacent to a more naïve strategy, namely, the *maturity strategy* and, that the portfolio design in regards to the immunization strategy matters to increase its efficiency.

The difference between the immunization model via $M$-vector and the one proposed in the present study is that in the first the immunization is based in a Taylor expansion which is the equivalent to a polynomial parametric model

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[6]The limitation of this approach is that its implementation demands a vast amount of securities, which is not observed in emerging market economies.

[7]The immunization is done using the security whose maturity is the closest in a investment horizon.

[8]*Maturity-bullet* is the strategy of combining the maturity with the closest security, but with greater maturity than the investment horizon. Maturity-barbell is the strategy of combining the security of the maturity strategy with a title of greater duration available in the sample. Both strategies must fulfill the restriction of same duration for the portfolio and investment horizon.

(when the higher orders in the Taylor expansion are used) and its implementation is hinged on the previous knowledge of the instantaneous forward rates. Additionally, the movements that the interest rate curve may suffer are not explicitly defined. The technique described in the present study, on the other hand, builds the curve movements using an exponential parametric model and proposes that the immunization be carried out by neutralizing these movements using a first order Taylor expansion. With that, the movements extracted are intuitive and very well defined, making their trading by fixed-income managers easier.

The procedure suggested in the present study generalizes the immunization based in principal components proposed by Litterman and Scheinkman (1991), because it does not need directly observable interest rates to build the latent orthogonal factors and, after that, use them for hedging. By choosing a parametric model, the factors that will be used for hedging result from the interest rate curve estimation process, regardless of being observable or unobservable. In fact, the immunization done using this approach is not only simple and efficient, but it is equivalent to the procedure proposed by Litterman and Scheinkman. The reason is that, according to Almeida et al. (2003), if a term structure model is separable; its factors are built by a rotation of the latent orthogonal factors that define the principal components.

Another contribution of our study is in presenting a methodology to build inflation curves, as well as comparing the inflation curve obtained from the Brazilian public debt bond market (LTN/NTN-F and NTN-B) with the one obtained from the derivatives market (DI Future- and IPCA Coupon [SDL]). This information is important, because it allows us to extract the expected future inflation implicit in instruments traded in the market. The future inflation measure thus extracted seems to be the most reliable one, since it is the result of bets on the future behavior of the real and nominal interest rates of the Brazilian economy by the market agents.

A third contribution of the present study is the definition of two sloping movements that can be used with the immunization approach based on *duration* and convexity to trade the slope of the interest rate curve. These definitions, although simple, are new, and highlight the fact that obtaining the slope and curvature movement within the *duration* and convexity framework is not a natural procedure. As a byproduct, the paper compares the proposed Svensson approach with the *duration* and convexity one to trade the level and slope of the interest rate curve (with and without the immunization of other risk factors that affect the interest rate curve).

This study is thus organized: section 2 presents the model, carries out an analysis of the model risk factors relating them to the principal components and shows how simple it is to price a fixed-income portfolio using that framework; section 3 discusses how to estimate the parameters of the model, and therefore, how to build a time series for the latent factors; section 4 presents the theory behind the

hedging of a fixed-income portfolio, as well as the procedures to trade on the curve movements, that is, on the latent factors of the model; section 5 shows examples of the method for hedging and leveraging in the Brazilian inflation-indexed public debt market and a comparison with the approach based on *duration* and convexity. Section 6 presents the methodology for the construction of the inflation curve and compares the curves extracted from two markets:

(1) the government bonds market and

(2) the derivatives market.

Finally, section 7 presents our conclusions.

## 2.  Models

### 2.1  Interest rate term structure

Nelson-Siegel (1987, 1988) originally derived the following model to describe the forward curve dynamic:

$$f\left(\tau\right) = a_1 + a_2 \exp\left(-\lambda_1\tau\right) + a_3\left(\lambda_1\tau\right)\exp\left(-\lambda_1\tau\right) \tag{1}$$

Using the following relation between the forward rates and the spot rate:

$$R\left(\tau\right) = \frac{1}{\tau}\int_0^\tau f\left(\tau\right)d\tau \tag{2}$$

It is possible to find the spot curve implied by the Nelson-Siegel model:

$$R\left(\tau\right) = b_1 + b_2\left(\frac{1-\exp\left(-\lambda_1\tau\right)}{\lambda_1\tau}\right) - b_3\exp\left(-\lambda_1\tau\right) \tag{3}$$

The following equation represents the model used by Diebold-Li (2006).

$$R\left(\tau\right) = a_1 + a_2\left(\frac{1-\exp\left(-\lambda_1\tau\right)}{\lambda_1\tau}\right) + a_3\left[\left(\frac{1-\exp\left(-\lambda_1\tau\right)}{\lambda_1\tau}\right) - \exp\left(-\lambda_1\tau\right)\right] \tag{4}$$

It is easy to notice there is a direct relation between (4) and (3): $b_1 = a_1, b_2 = a_2 + a_3$ and $b_3 = a_3$. Diebold and Li modified the original model by Nelson-Siegel (1987), because the loadings of $b_2$ and $b_3$ have very similar decay forms, which makes it difficult to provide an intuitive interpretation of the factors, as in Litterman and Scheinkman of level, slope and curvature. Additionally, it poses the problem of multicolinearity in regards to the estimation of the factors, due to their similarity, which could affect the precision of the estimation.

The Diebold-Li model is also asymptotically interesting, because it generates a discount function that starts in one for maturity zero and converges to zero in

infinity. Such characteristic causes the obtained curve to conform better to the rates of the longer maturity securities.

By adding a new curvature (U-shaped) to the equation (4) an extension of the Diebold-Li model proposed by Almeida et al. (2007) is obtained. The new parameter is important for the modeling of interest rate curves in highly volatile markets, such as the Brazilian market, because it allows a greater flexibility of the curve to capture more complex and non-linear movements resulting from the excess volatility in the rates. The model relates continuously composed spot rates with risk factors and loadings through the following function:

$$
\begin{aligned}
R\left(\tau\right) \quad = \quad & a_1 + a_2 \left(\frac{1 - \exp\left(-\lambda_1\tau\right)}{\lambda_1\tau}\right) + a_3 \left[\left(\frac{1 - \exp\left(-\lambda_1\tau\right)}{\lambda_1\tau}\right) - \exp\left(-\lambda_1\tau\right)\right] \\
& + a_4 \left[\left(\frac{1 - \exp\left(-\lambda_2\tau\right)}{\lambda_2\tau}\right) - \exp\left(-\lambda_2\tau\right)\right]
\end{aligned} \quad (5)
$$

onde $R\left(\tau\right)$ is the continuously composed spot interest rate of a zero coupon bond with maturity $\tau$;

$a_1$, $a_2$, $a_3$ and $a_4$ are risk factors regarding level, slope and curvature;

$\lambda_1$ and $\lambda_2$ are parameters of scale.

The parameters of scale govern the exponential decay: low values produce a slow decay and better adjust the curve to the longer maturities.

Additionally, the $\lambda$'s govern the horizon period in which the loadings on $a_3$ and $a_4$ reach their peak. Because both loadings are U-shaped, choosing the maturity $(\tau)$ that maximizes the loading is equivalent to solving the

$$
\exp\left(-\lambda\tau\right) + \frac{\exp\left(-\lambda\tau\right)}{\lambda\tau} - \frac{\left(1 - \exp\left(-\lambda\tau\right)\right)}{\left(\lambda\tau\right)^2} = 0 \quad (6)
$$

The loadings on the coefficients $a_1$, $a_2$, $a_3$ and $a_4$ can be interpreted as the sensitivity of the spot rate in relation to the variation of the latent factors, therefore, can be obtained using the first derivative in relation to each of the risk factors:

$$
\begin{aligned}
\frac{\partial R\left(\tau\right)}{\partial a_1} \quad &= \quad 1 \\
\frac{\partial R\left(\tau\right)}{\partial a_2} \quad &= \quad \left(\frac{1 - \exp\left(-\lambda_1\tau\right)}{\lambda_1\tau}\right) \\
\frac{\partial R\left(\tau\right)}{\partial a_3} \quad &= \quad \left[\left(\frac{1 - \exp\left(-\lambda_1\tau\right)}{\lambda_1\tau}\right) - \exp\left(-\lambda_1\tau\right)\right] \\
\frac{\partial R\left(\tau\right)}{\partial a_4} \quad &= \quad \left[\left(\frac{1 - \exp\left(-\lambda_2\tau\right)}{\lambda_2\tau}\right) - \exp\left(-\lambda_2\tau\right)\right]
\end{aligned} \quad (7)
$$

The loadings as a function of the maturity can be seen in Figure 1. The loading on $a_1$ is constant and can be seen as a long-term equilibrium rate, because it does

not decay to zero in maturities that tend to infinite. The loading on $a_2$ starts at one and quickly decays to zero, therefore, it has a greater influence on short-term interest rates. The loadings on $a_3$ and $a_4$ start at zero, increase, and then decay to zero, therefore they can been seen as influencing the medium-short and medium-long parts of the curve, respectively. That is, they influence the curve's convexity. They are similar to the loadings obtained using Litterman-Scheinkman (1991) principal components analysis, and therefore can be interpreted as: level, slope, first curvature and second curvature, respectively.

Figure 1
Loadings of the real interest rate curve (NTN-B) with $(\lambda_1, \lambda_2) = (0.3, 0.2)$
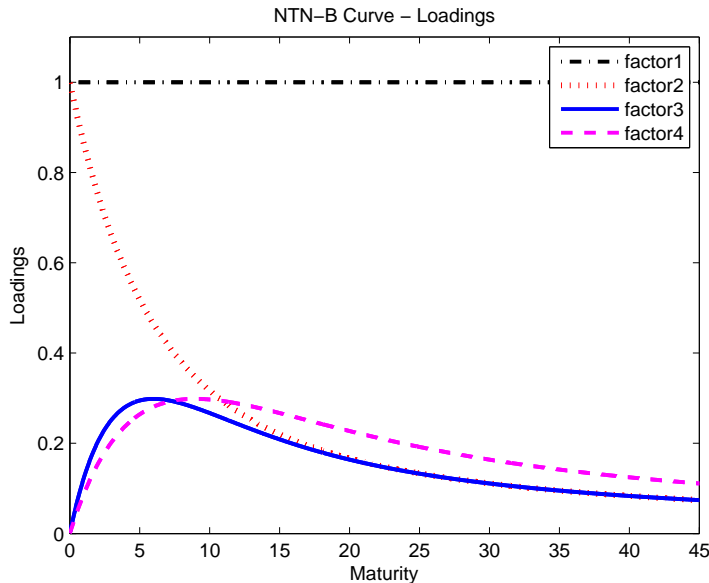


As it can be seen in the correlation matrix below, the factors of the models are non-orthogonal, as in Scheinkman-Litterman, that is, they correlate to each other.[9]

$$\begin{bmatrix} 1 & 0.09 & -0.14 & -0.35 \\ & 1 & -0.71 & 0.45 \\ & & 1 & -0.83 \\ & & & 1 \end{bmatrix}$$

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[9]For statistical analysis of this section we have used the daily data of the Brazilian IPCA-indexed securities known as NTN-B's supplied by Andima. The data window goes from 10-19-2004 to 5-15-2007, with a total of 644 observations. The parameters of scale used for the construction of the curves were $(\lambda_1, \lambda_2) = (0.3, 0.2)$.

Notice, specifically, the high negative correlation between $a_3$ and $a_4$ and, also the relation between $a_2$ and $a_3$. These facts can be better observed in Figure 2, which shows the evolution of the factors through time.

It is possible to obtain the orthogonal structure of the factors that would be generated by the interest rates using the principal components analysis directly on the series of factors, because according to Almeida et al. (2003), the latent factors obtained from the principal components analysis of the rates are merely a rotation of the model's own factors. Therefore, two steps are necessary to obtain the loading of the factors that would be obtained by applying the principal components directly on the rates:

(1) apply the principal components analysis directly on the parameters series of the model and

(2) weigh the loadings of the model by the coefficient matrix found in step 1.

Step two is necessary because the coefficient matrix describes only the participation of each factor/parameter of the model for the construction of the principal components. The following graph shows the sensitivity of the rate in regards to a shock in the principal components.

Figure 2
Evolution of the latent factors in the real interest rate (NTN-B)
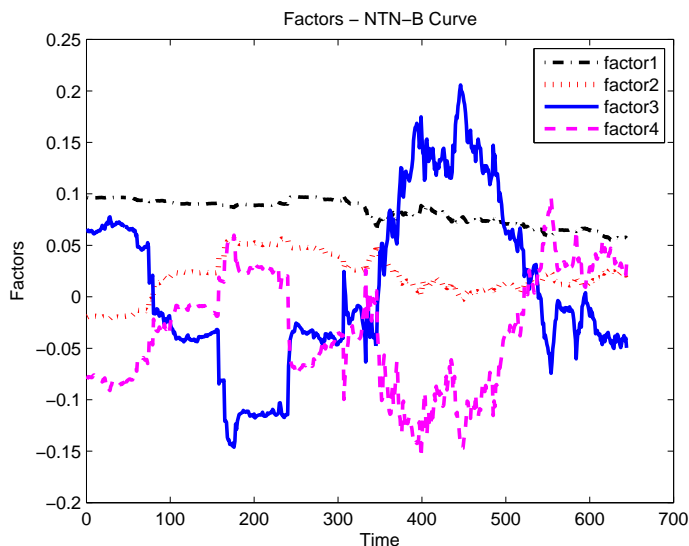


To better interpret the orthogonal factors it is worth observing the coefficient

matrix used in step 2 above:

$$
\begin{bmatrix}
-0.05 & 0.89 & -0.34 & 0.3 \\
0.52 & 0.25 & 0.79 & 0.21 \\
-0.62 & -0.16 & 0.28 & 0.71 \\
0.57 & -0.33 & -0.43 & 0.61
\end{bmatrix}
$$

This matrix shows how the orthogonal structure is built. The loading on the first factors has an interpretation of level, because it affects the whole curve positively, although the rates in the beginning of the curve are more affected, and the effect lessens until maturity five. The loadings on the second factor also have an interpretation of level, because it affects the whole curve positively, despite falling monotonically until maturity five and presenting a slight increase after that. The loading on the third factors has a clear interpretation of slope, because it decays monotonically crossing the zero axes after six years (which represents a movement of reduction in the interest rate curve slope). The loading on the last factor can be clearly interpreted as curvature, because despite the fact that it positively affects all maturities in the curve, the effect is much stronger in median maturities.

Using the principal component analysis is interesting because it allows us, through the eigenvalues, to know how much each component explains of the total movements of the curve. The first component is responsible for 58.68% of the total movements of the curve; the second is responsible for 29.55%, while the third and fourth explain 11.38% and 0.39%, respectively. It seems that the addition of the fourth factor does not help explain the movements of the curve, but as discussed in Almeida et al. (2007), the inclusion increases the ability to generate more volatile and non-linear interest rate curves, by modifying the risk premium structure, and it also improves the forecasting capacity of the model for shorter maturities in comparison to the Diebold and Li model (which does not include this factor).

## 2.2 Pricing of fixed-income portfolios

The price of any security can be found by discounting its coupons by the rate relative to the maturity of that cash flow:

$$
P_j = \sum_{t_i}^{T^j} c^j(t_i) \exp\left(-R(t_i) \times t_i\right) \tag{8}
$$

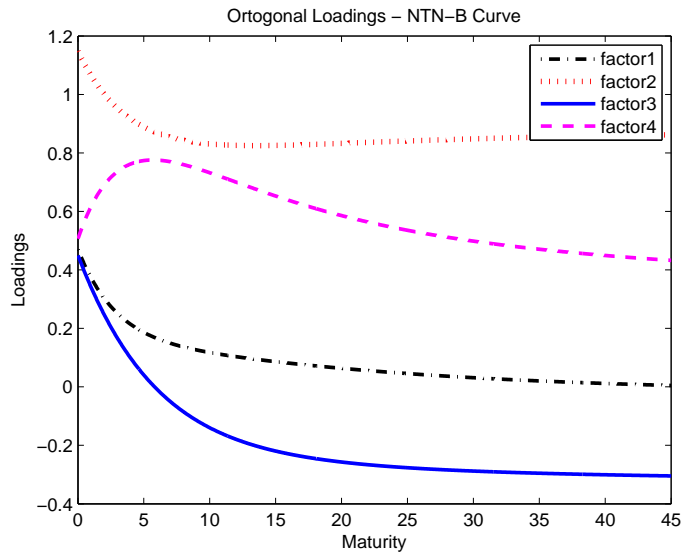where: $P_j$ is the price of the security $j$ of a portfolio.
$c^j(t_i)$ is the coupon $i$ of security $j$ that will be paid on time $t_i$.
$i$ is the number of coupons of security $j$ from today to maturity.
$T^j$ is the maturity of the security $j$.

Then, the price of a portfolio $j = 1, 2, \ldots, J$ securities with weighs,

Figure 3
NTN-B orthogonal curve loadings



$\omega_1, \omega_2, \ldots, \omega_j$, so that $\sum_j \omega_j$; can be described as:

$$
\begin{aligned}
P &= \sum_j \omega_j \sum_{t_i}^{T^j} c^j(t_i) \exp(-R(t_i) \times t_i) \\
&= \sum_{t_i}^{MaxT^j} \left( \sum_j \omega_j c^j(t_i) \right) \exp(-R(t_i) \times t_i) \\
&= \sum_{t_i}^{MaxT^j} c(t_i) \exp(-R(t_i) \times t_i) \\
&= \sum_{t_i}^{T} c(t_i) \exp(-R(t_i) \times t_i)
\end{aligned}
\tag{9}
$$

Therefore, any portfolio with $J$ securities can be seen as one security with distinct and non-uniform coupons throughout time.

The next section shows how the parameters of the model can be estimated using a group of J securities in a given moment in time.

## 3. Estimation

The first step is to estimate the coefficients $a_{t,1}, a_{t,2}, a_{t,3}, a_{t,4}, \lambda_{t,1}, \lambda_{t,2}$, for each instant in time $t$, since we are interested in the evolution of the term structure throughout time. Such dynamics can be seen as a panel with the following axes: maturity of the securities and coupon payments, and time. The equation (5) can be rewritten to incorporate the dynamic and statistical aspects of the model as such:

$$
\begin{aligned}
R_t(\tau) \; = \; & a_{t,1} + a_{t,2}\left(\frac{1 - \exp(-\lambda_{t,1}\tau)}{\lambda_{t,1}\tau}\right) && (10) \\
& + a_{t,3}\left[\left(\frac{1 - \exp(-\lambda_{t,1}\tau)}{\lambda_{t,1}\tau}\right) - \exp(-\lambda_{t,1}\tau)\right] \\
& + a_{t,4}\left[\left(\frac{1 - \exp(-\lambda_{t,2}\tau)}{\lambda_{t,2}\tau}\right) - \exp(-\lambda_{t,2}\tau)\right] + \varepsilon_{t,\tau}
\end{aligned}
$$

Let us assume that there are $J$ securities available in the instant $t$. The estimation of the coefficients is done by minimizing some loss function. Ideally, the loss function should involve observed rates and rates from the model. However, this is not always possible; for example, securities that do have coupons do not have observable rates. In this case, the loss function must be built using the observed price and the model price. Examples of this type of loss functions are: sum of the price squared residuals, sum of the price absolute residuals, and weighted sum of the price squared residuals, among others. This last approach is interesting because it allows us to give more weight to the most important securities in a specific market in order to avoid distortions on the estimated interest rates caused by low liquidity or maturity. Using the weighted sum of the price squared residuals as a loss function, the parameters of the model are obtained by solving the following optimization problem (non-linear generalized minimum squares).

$$
Min_{a_{t,1}, a_{t,2}, a_{t,3}, a_{t,4}, \lambda_{t,1}, \lambda_{t,2}} \sum_{j=1}^{J} w_{t,j}\left(\overline{P_{t,j}} - P_{t,j}\right)^2 \qquad (11)
$$

$P_{t,j}$ is the price of the security j given by equation (8) of the model in the period of time $t$;
$\overline{P}_{t,j}$ is the observed market price of the security $j$ in the period of time $t$;
$\omega_{t,j}$ is the weight which attributes the degree of importance of the security $j$ in the period of time $t$.

In order to use the model for fixed-income operations, it is necessary to estimate it daily, in a way it is possible to build a time series for the coefficients: $a_{t,1}, a_{t,2}, a_{t,3}, a_{t,4}, \lambda_{t,1}, \lambda_{t,2}$, for each $t = 1, 2, \ldots, T$.

It is important to mention that the scale parameters $(\lambda_{t,1}, \lambda_{t,2})$ are very instable, that is, the sequential estimation finds very different parameters throughout

time. It is worth mentioning that depending of the application it is important to fixate them in time $(\lambda_{t,1}, \lambda_{t,2}) = (\lambda_1, \lambda_2)$ for every $t$. Some examples of applications when fixation is interesting are: forecasting of future rates, directional trading of the interest rate curve, and for hedging of fixed-income portfolio.

When the rates are observable, as is the case of most interest rate derivatives with liquidity in Brazil, it is interesting to use a loss function that deals with rates instead of prices. There are two main reasons for this change in approach:

(1) to avoid distortions caused by the maturity of the instrument[10] and

(2) if the parameters of scale are constant, the estimation of the model can be done sequentially using linear regression or GLS, reducing the computational cost and increasing reliability of the estimate (assurance of global optimum).

In this way the problem given in equation (11) is modified as it follows:

$$\min_{a_{t,1}, a_{t,2}, a_{t,3}, a_{t,4}, \lambda_{t,1}, \lambda_{t,2}} \sum_{j=1}^{J} w_{t,j} \left( \overline{y_{t,j}} - y_{t,j} \right)^2 \qquad (12)$$

If the parameters of scale are constant, the problem is reduced to:

$$\min_{a_{t,1}, a_{t,2}, a_{t,3}, a_{t,4}} \sum_{j=1}^{J} w_{t,j} \left( \overline{y_{t,j}} - y_{t,j} \right)^2 \qquad (13)$$

$y_{t,j}$ is the continuously composed annual rate of the security $j$ given by the equation (8) of the model in the period of time $t$;
$\overline{y}_{t,j}$ is the continuously composed annual rate observed in the security market of the security $j$ in the period of time $t$.

For hedging applications, curve movement trading and construction of factors we used the procedure given by equation (11), because those will be based on the NTN-B's. For comparison between breakeven inflation in the domestic public debt securities (LTN/NTN-F and NTN=B) and in the interest rate derivatives (DI-futuro and IPCA coupon) both types of loss function (11) and (13) will be used.

### 3.1 Data

For the analysis carried out in the section Simulation we used the daily data of the Brazilian IPCA-indexed public debt bonds, known as NTN-B's. The data

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[10]The prices of securities about to mature vary very little even when the rates present great variations. For example, the price of a security with maturity of 1 day which pays one on the maturity varies from 0.9996 to 0.9993 when the rates are 10%pa and 20%pa, respectively (under the day counting rule: du/25). That is, an error of 1000 bps in rate is equivalent to an error of 0.0003 price monetary units.

window goes from 10-19-2004 to 5-15-2007, in a total of 644 observations. Each observation (day) of the sample has a group of NTN-B bonds with the following characteristics: reference date, maturity, yield to maturity (expectation), PU (bond price), VNA (current nominal value), Quotation (price of a security with face value 1 and the same characteristics) and Coupon[11] (as a percentage of face value). The following tables show, respectively, the first and last observations of the sample in the time dimension (dates 10-19-2004 and 05-15-2007):

| Reference date | Maturity | Yield to maturity | PU | VNA | Quotation | Coupon |
|---|---|---|---|---|---|---|
| 15/05/2007 | 15/08/2008 | 7,5457 | 1.639,23949 | 1.644,04008 | 0,997080 | 6% aa |
| 15/05/2007 | 15/05/2009 | 7,0148 | 1.614,67588 | 1.644,04008 | 0,982139 | 6% aa |
| 15/05/2007 | 15/11/2009 | 6,9935 | 1.608,22138 | 1.644,04008 | 0,978213 | 6% aa |
| 15/05/2007 | 15/08/2010 | 6,7352 | 1.634,65919 | 1.644,04008 | 0,994294 | 6% aa |
| 15/05/2007 | 15/05/2011 | 6,5612 | 1.613,93935 | 1.644,04008 | 0,981691 | 6% aa |
| 15/05/2007 | 15/08/2012 | 6,4645 | 1.635,78371 | 1.644,04008 | 0,994978 | 6% aa |
| 15/05/2007 | 15/11/2013 | 6,4075 | 1.609,94927 | 1.644,04008 | 0,979264 | 6% aa |
| 15/05/2007 | 15/05/2015 | 6,2645 | 1.618,87969 | 1.644,04008 | 0,984696 | 6% aa |
| 15/05/2007 | 15/05/2017 | 6,2576 | 1.615,52914 | 1.644,04008 | 0,982658 | 6% aa |
| 15/05/2007 | 15/03/2023 | 6,2425 | 1.623,93676 | 1.644,04008 | 0,987772 | 6% aa |
| 15/05/2007 | 15/08/2024 | 6,2087 | 1.635,57492 | 1.644,04008 | 0,994851 | 6% aa |
| 15/05/2007 | 15/11/2033 | 6,1880 | 1.608,25591 | 1.644,04008 | 0,978234 | 6% aa |
| 15/05/2007 | 15/05/2035 | 6,1729 | 1.610,94062 | 1.644,04008 | 0,979867 | 6% aa |
| 15/05/2007 | 15/05/2045 | 6,1488 | 1.613,02362 | 1.644,04008 | 0,981134 | 6% aa |

| Reference date | Maturity | Yield to maturity | PU | VNA | Quotation | Coupon |
|---|---|---|---|---|---|---|
| 19/10/2004 | 15/08/2006 | 8,6518 | 1.411,06971 | 1.456,57802 | 0,968757 | 6% aa |
| 19/10/2004 | 15/05/2009 | 8,7596 | 1.352,09959 | 1.456,57802 | 0,928271 | 6% aa |
| 19/10/2004 | 15/11/2013 | 8,7625 | 1.254,51531 | 1.456,57802 | 0,861276 | 6% aa |
| 19/10/2004 | 15/05/2015 | 8,8025 | 1.226,95212 | 1.456,57802 | 0,842352 | 6% aa |
| 19/10/2004 | 15/03/2023 | 8,8499 | 1.103,35491 | 1.456,57802 | 0,757498 | 6% aa |
| 19/10/2004 | 15/08/2024 | 8,8684 | 1.097,32954 | 1.456,57802 | 0,753361 | 6% aa |
| 19/10/2004 | 15/11/2033 | 8,8979 | 1.069,09695 | 1.456,57802 | 0,733979 | 6% aa |
| 19/10/2004 | 15/05/2045 | 9,0997 | 1.022,68791 | 1.456,57802 | 0,702117 | 6% aa |

It is important to point out that for a given period of time there is also observations in the cross-section dimension (bonds of different maturities); therefore the sample has characteristics of a non-balanced panel, because observations in the cross-section dimension vary with time.

Observe the column Expectation in both tables and notice how the real interest rate curve changed the level. Take as an example the security with maturity in the year 2045. Its rate in the beginning was 9.1%, while at the end of the sample

---

[11]NTN-B has a coupon of 6% pa with biannual payments calculated as:

$$\left[(1 + 6\%)^{1/2} - 1\right] \times VNA$$

.

it had fallen to 6.15%.

To better understand the data it is worth mentioning that the NTN-B can be decomposed in two parts:

(1) the part that accumulates the inflation rate daily (VNA) and

(2) the discount part (Quotation).

By multiplying these two parts we get the Bond Price (PU). More details on the NTN-B can be found directly in the National Treasury website (www.tesourodireto .gov.br).

For the analysis carried out in the section implicit expectations extractions, we used the daily data of the fixed-income Brazilian public debt bonds, known as LTN's and NTN-F's, and IPCA-indexed (NTN-B's). Additionally, we used daily data of the 1 day interbank rate future contracts (referred to as DI-Futuro) and of the IPCA-DI swap contracts (registered at BM&F as SDL, and from now on referred as IPCA coupon), both contracts are registered at the BM&FBOVESPA (Securities, Commodities and Futures Exchange).

The first derivative contract is standardized, and, therefore, actively traded at BM&FBOVESPA, its holder receives the amount of money resulting from the difference between a fixed-income rate and the daily accumulated 1 day interbank rate (DI); the IPCA coupon is registered and traded over the counter, and has a much lower liquidity then DI-Futuro. Its holder received the amount of money resulting from the difference between a real fixed-income rate plus accumulated inflation and the daily accumulated 1 day interbank rate (DI).

The data for the section Extraction of implicit expectations ranges from 1-2-2006 to 5-12-2007, with 21 observations in total. The reduced number of observations is caused by the IPCA coupon for two reasons:

(1) its moderate liquidity and

(2) the exclusion from the sample of days in which four or less contracts were traded.

This exclusion is necessary because of the estimation process, to estimate 4 parameters a minimum of 4 observations are needed.

## 3.2 Parameter of scale

In the section Simulation the parameters of scale $(\lambda_{t,1}, \lambda_{t,2})$ were fixed in the following manner. First, all parameters were estimated daily (with $w_{t,j} = 1$, for every $t$ and every $j$) with a reduced window of 300 observations – from 3-2-2006 to 5-15-2007 – and the average in time from $(\lambda_{t,1}, \lambda_{t,2})$ was calculated, in a way that the values found for the NTN-b curve were (0.3, 0.2).[12] Given

---

[12]This means that the highest points of both curves, according to the equation 6, are, respectively, 5.98 and 8.96 years.

the fixed values $(\lambda_1, \lambda_2)$, the model was reestimated in the original window to find $(a_{t,1}, a_{t,2}, a_{t,3}, a_{t,4})$. The idea behind using a reduced window to find the parameters of scale is to try to give more weight to the more recent observations in the time dimension in order to obtain a better characterization of the current interest rate curve.

Alternative procedures can be used to fixate the parameters of scale. Diebold and Li (2006) chose the parameter of scale in a way that the highest point of the curve was the 30 month maturity. The choice of this maturity is done by taking the average between two maturities usually chosen for this question (24 and 36 months). Another procedure would be to build a grid of possible parameters of scale, estimate the model using the parameters of scale in each one of the entries in the grid and choose the one with the lowest value for the loss function. The problem of this last approach is that the parameters of scale could possibly vary from one day to the next (every time the model was reestimated), making the factors of the model, except for the first one, impossible to compare. This last point can be understood by observing equation (7) and noticing that the loadings of the model are affected by the parameters of scale. Despite the loss in the intertemporal comparison dimension, the grid procedure allows for better adjustment of the model to the observable data, that is, a lower loss function.

Any of the procedures described above can be used to build the time series of the factors of any interest rate curve, and specially, the four types of interest rate curves of the Brazilian public debt with the greatest liquidity:

(1) fixed-income (NTN-F and LTN),

(2) obtained by bonds that are indexed to the IPCA (NTN-B),

(3) obtained by the bonds that are indexed to the IGP-m (NTN-C) and

(4) the one obtained by the bonds indexed by exchange rate variations (NTN-D).

As well as for interest rate curves originated from derivative contracts: DI-Futuro, IPCA coupon, IGP-m coupon and exchange rate coupon, among others.

For the section Extraction of implicit expectations, we have chosen to use the grid procedure for two reasons:

(1) to illustrate the approach and

(2) we believe that the adjustment of the model to the data for this analysis is more important than the comparability of the factors through time.

### 3.3   Results of the estimation

To illustrate the dynamic of factors, the following graph shows the time series of the four latent factors of the NTN-B curve when $(\lambda_{t,1}, \lambda_{t,2})$ are fixed at (0.3,

Figure 4
Evolution of factors for NTN-B curve
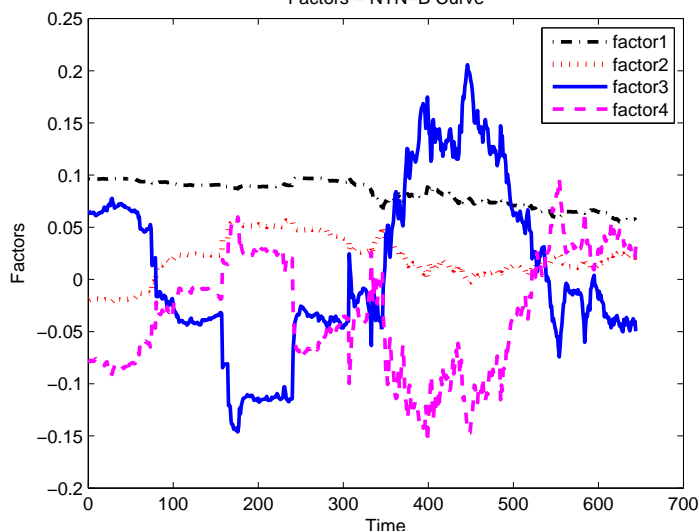


0.2). Observe in Figure 4 the inverse behavior of the convexities (long and short), as well as the declining tendency of the first factor (level). The fact that the level of the real interest rate curve is declining with time can be understood as a reflex of the macroeconomic policies adopted in the recent past: fiscal adjustment (primary surplus target), flexible exchange rate, inflation target regime, change in the government debt profile, among others.

Observe the shape of the real interest rate curve implied from the NTN-B's of the last day of the sample in Figure 5. Notice that the curve clearly embeds the expectation of the market agents of continuing decline in the Brazilian real interest rate.

It is important to highlight that, given the characteristics of the NTN-B bonds of having intermediate coupons; it is not possible to graphically compare the curve in Figure 5 to the market curve, because, as discussed in the Introduction, the interest rates in that market are unobservable. The solution to know how well the curve is adjusted to the dada is through the residuals from some type of loss function already described in the beginning of the Estimation section. Figure 6 describes the evolution of the percentage error of security-$j$ price in the period of time $t$. This is defined as:

$$ep_{t,j} = \frac{\left( \overline{P_{t,j}} - P_{t,j} \right)}{P_{t,j}} = \frac{e_{t,\tau_j}}{P_{t,j}} \tag{14}$$

where $\tau_j$ is the maturity of the security $j$ and $e_{t,\tau_j}$ is the pricing error of the security with maturity $\tau_j$.

Figure 5
Real interest rate curve, on 05-15-2007, implied from the NTN-B's
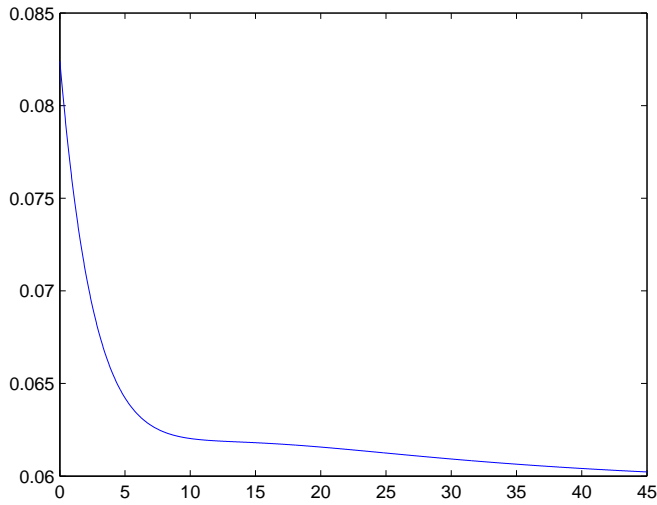


Notice that the pricing errors in the model are, at most 2%, despite being lower than 1% for the largest part of the sample (and most of the securities).

Given the time factors series it is possible to build a tri-dimensional graph – Figure 7 – of the evolution of the NTN-B's interest rate curve.

Notice that the level of the curve falls with time and the slope of the curve at the beginning of the sample is positive, it inverts after a few periods and remains negative until the end of the sample. This movement can be interpreted as a gradual reduction of the real interest rate and the expectations of the future real interest rate.

## 4. Model analysis: hedging and leveraging

Let $P$ be a portfolio of $J$ securities given by:

$$P = \sum_{t_i}^{T} c\left(t_i\right) \exp\left(-R\left(t_i\right) \times t_i\right) \tag{15}$$

## 4.1 First order conditions

Because the rate given by any parametric model can be described as a function of the risk factors (coefficients), $R(\tau) = R(\tau, a_1, a_2, a_3, a_4)$ it is easy to see that the sensitivity in the value of a portfolio generated by (infinitesimal) changes in the coefficients is given by:

$$\frac{\partial P}{\partial a_1} = \sum_{t_i}^{T} -t_i c(t_i) \exp(-R(t_i) \times t_i) \frac{\partial R(t_i)}{\partial a_1}$$

$$\frac{\partial P}{\partial a_2} = \sum_{t_i}^{T} -t_i c(t_i) \exp(-R(t_i) \times t_i) \frac{\partial R(t_i)}{\partial a_2} \qquad (16)$$

$$\frac{\partial P}{\partial a_3} = \sum_{t_i}^{T} -t_i c(t_i) \exp(-R(t_i) \times t_i) \frac{\partial R(t_i)}{\partial a_3}$$

$$\frac{\partial P}{\partial a_4} = \sum_{t_i}^{T} -t_i c(t_i) \exp(-R(t_i) \times t_i) \frac{\partial R(t_i)}{\partial a_4}$$

where $\left( \frac{\partial R(t_i)}{\partial a_1}, \frac{\partial R(t_i)}{\partial a_2}, \frac{\partial R(t_i)}{\partial a_3}, \frac{\partial R(t_i)}{\partial a_4} \right)$ is given by (7).

These derivatives are named *duration*, because they inform how much the portfolio varies financially with increases of one unit in each of the risk factors. It is important to point out that these derivatives do depend on the loadings

$$\left( \frac{\partial R(t_i)}{\partial a_1}, \frac{\partial R(t_i)}{\partial a_2}, \frac{\partial R(t_i)}{\partial a_3}, \frac{\partial R(t_i)}{\partial a_4} \right)$$

evaluated in each maturity $t_i$, that is on the interest rate curve obtained through the parametric model $R_{(ti)}$.

A simple Taylor expansion of first order around the initial parameters shows how the value of a portfolio is related to the *duration*.

$$\Delta P = P(a) - P(a_0) = D_a P \cdot (\Delta a) + erro(o) \qquad (17)$$

where,
$P(a)$ is the price of the portfolio when the factors are given by the vector $a$;
$D_a P$ is the gradient vector $\left( \frac{\partial P}{\partial a_1}, \frac{\partial P}{\partial a_2}, \frac{\partial P}{\partial a_3}, \frac{\partial P}{\partial a_4} \right)$;
$(\Delta) a$ is the difference vector of the parameters$(a - a_0)$;
*error* $(o)$ is the approximation error for orders higher than 1.

Figure 6

Evolution of percentage errors: Difference between market price and model price
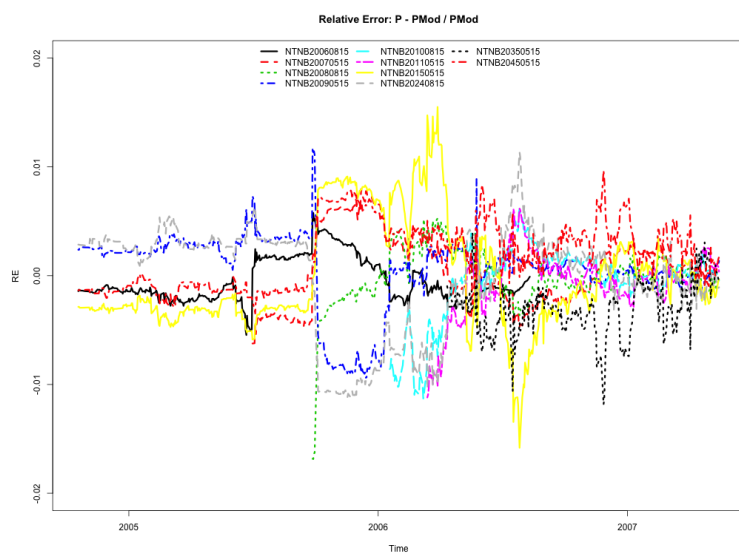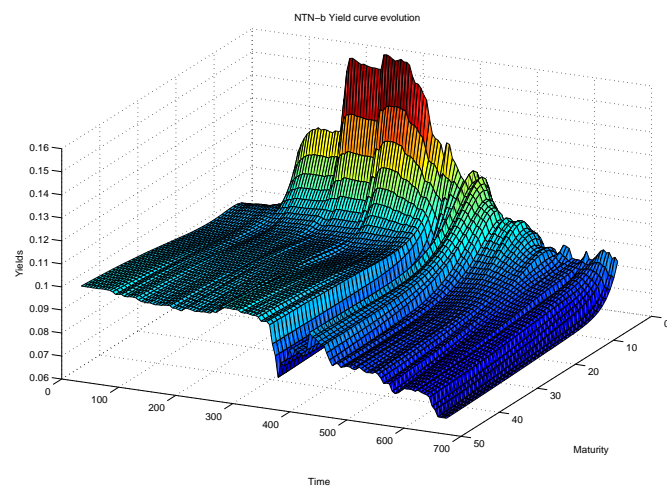divided by model price



Figure 7

Evolution of NTN-B Curve

## 4.2 Hedging a single risk factor

Suppose that an agent holds a portfolio $P$ for which he or she would like to eliminate the risk generated by variations in the risk factors $a_i$, $i = 1, 2, 3, 4$.

To do that, the agent needs to build a portfolio $G$ composed by 1 unit of $P$ and $\phi$ units of an instrument $H$ (a security or a derivative).

That is,

$$G = P + \phi H \tag{18}$$

The amount $\phi$ must be such those small movements in do not cause profits or losses in G:

$$\frac{\partial G}{\partial a_i} = \frac{\partial P}{\partial a_i} + \phi \frac{\partial H}{\partial a_i} = 0 \tag{19}$$

$$\phi = -\frac{\frac{\partial P}{\partial a_i}}{\frac{\partial H}{\partial a_i}} \tag{20}$$

where $\phi H$ is the financial amount on the instrument needed to hedge the portfolio for variations in $a_1$.

## 4.3 Leveraging a single risk factor

The same analysis can be done if the agent desires to increase the exposition of $P$ to the risk factor $a_i$, for example, by multiplying it by $x$.

Similarly to the previous case, an agent needs to build a portfolio $G = P + \phi H$ so that:

$$\phi = \frac{\frac{\partial P}{\partial a_i}}{\frac{\partial H}{\partial a_i}} (x - 1) \tag{21}$$

## 4.4 Hedging or leveraging: multiple risk factors

Suppose there are $N$ risk factors. It is possible to hedge one factor, while the exposure to the others is increased, or to increase the exposure of one factor while leaving the exposure to the others approximately fixed. For example, to trade the level of a curve and at the same time neutralizes the exposure to changes in slope and convexity of the curve. Such operations can be interesting to take advantage of a specific movement in the market that the manager expects will come in a few periods of time. For example, recently in Brazil, there was a pronounced decrease in the domestic yield rates of NTN-B's.

The analysis used before can be done in this more general case. To this end, all that is needed is to observe the following rule of thumb: the number of instruments must be the same as the number of factors one wishes to trade (hedging or leveraging). Therefore, if there are $N$ risk factors, then $N$ instruments are necessary.

Let $H_1, H_2, ..., H_N$ be instruments. And suppose that the agent has a portfolio $P$ and wants to increase the exposure to factor 1 $x_1$ times, factor 2 $x_2$ times, ...,

and factor N $x_N$ times. To do so, one needs to build the augmented portfolio $G = P + \phi_1 H_1 + \phi_2 H_2 + ... + \phi_N H_N$ so that:

$$\frac{\partial G}{\partial a_i} = \frac{\partial P}{\partial a_i} + \phi_1 \frac{\partial H_1}{\partial a_i} + \phi_2 \frac{\partial H_2}{\partial a_i} + ... + \phi_N \frac{\partial H_N}{\partial a_i} = \frac{\partial P}{\partial a_i} x_i \tag{22}$$

for every $i = 1, 2, \ldots, N$
or in matrix notation

$$(D_a G)^T = (D_a P)^T + (D_a H)^T \phi = diag \left[ (D_a P)^T \right] x \tag{23}$$

Leveraging and hedging consist in solving a system of linear equations, where the solution is the vector of amount of instruments to be purchased/sold.

$$\phi = \left[ (D_a H)^T \right]^{-1} diag \left[ (D_a P)^T \right] (x - \iota) \tag{24}$$

The matrix $(D_a H)^T$ must be non-singular so that its inverse $\left[ (D_a H)^T \right]^{-1}$ is well defined. Usually, in empirical applications, the matrix $(D_a H)^T$ is always invertible.

If $x = 0$, the case of complete immunization, then the solution is reduced to:

$$\phi = - \left[ (D_a H)^T \right]^{-1} (D_a P)^T \tag{25}$$

Another very useful concept to operate the curve movements in a way that the initial investment is zero is the self-financing portfolio. That is, buying and selling instruments so that your portfolio (of instruments) has value zero.

$$\phi \cdot H = 0 \tag{26}$$

To incorporate this restriction in the set of equations above it is necessary to include one more financial instrument. In this manner, the final system will have only a unique solution (given that the equations are not linearly dependents).

## 5. Simulation

In this section we present two examples of the developed methodology:

(1) directional level trading and

(2) slope directional trading with immunization of the other curve movements, as well as a comparison with several operational procedures created using the duration and convexity approach.

The convexity and duration approach is based on the Taylor approximation of first and second order, respectively, for the price of a fixed-income portfolio in relation to the yields to maturity (YTM) of each instrument in the portfolio. Two steps are necessary to immunize a portfolio based in duration and convexity:

(1) calculate the first and second order derivatives of the prices of the securities in the portfolio in relation to their yields to maturity;

(2) build a portfolio ($\phi$) so that the first and second derivatives such that the yields are zero.

Usually, it is imposed by hypothesis that the crossed derivatives are negligible, and therefore, zero. For more details, see Fabozzi (2001) and Martellini et al. (2003).

This section presents a new contribution to the literature by presenting two definitions of slope movements to be used with the duration/convexity approach, namely:

(1) The inverse of duration, "$D \times 1/D$", and

(2) Svensson Loading, "$D \times SV\_L2$".

## 5.1 Example 1: Level directional trading

Let us assume that on 07-25-2006 it is expected that the level of the NTN-B curve will rise. Suppose that there are only two IPCA-indexed bonds in the Brazilian public securities market:

1. NTN-B with short maturity 08-15-2010, biannual coupon payments of 6%pa and price and yields given by $(P_1, y_1) = (0.8929, 0.0979)$;

2. NTN-B with long maturity 05-15-2045, biannual coupon payment of 6%pa and price and yield given by $(P_2, y_2) = (0.7484, 0.0802)$.

The estimated coefficients represent the different movements of the interest rate curve present the following values:

$$a0 = (a_1, a_2, a_3, a_4) = (0.075, 0.0036, -0.1226, 0.1781)$$

To calculate the sensitivities, suppose that the face value of the bond is 1.[13] Therefore, the price and *duration* in relation to the $a_1$ factor and the yields $(y_1, y_2)$

---

[13]To obtain the real price and sensitivity values we must multiply the price and *duration* vectors by the Updated Nominal Values (UMV) in a way that the ratio of securities does not change.

are given by:

$$
\begin{aligned}
(P_1, P_2) &= (0.8929, 0.7484) \\
\left( \frac{\partial P_1}{\partial a_1}, \frac{\partial P_2}{\partial a_1} \right) &= (-3.1291, -9.9610) \\
\left( \frac{\partial P_1}{\partial y_1}, \frac{\partial P_2}{\partial y_2} \right) &= (-3.1313, -9.2718)
\end{aligned}
$$

Given the expectation of increase in the level factor, assume a self-financing directional position bought in level in such way that 1 unit of increase in $a_1$ increases the value of the portfolio in, for example, 1MM. Two instruments are needed to carry out the operation: one to leverage the level of the curve and another to build the self-financing portfolio.

### 5.1.1 Svensson Methodology

The solution to the problem is buying the short security and selling the long one in amounts $(\phi_1, \phi_2)$ given by the system bellow:

$$
\begin{aligned}
(\phi_1, \phi_2) \cdot \left( \frac{\partial P_1}{\partial a_1}, \frac{\partial P_2}{\partial a_1} \right) &= 1MM \\
(\phi_1, \phi_2) \cdot (P_1, P_2) &= 0
\end{aligned}
\tag{27}
$$

which has the solution:

$$
\phi = \left[ \begin{array}{cc} \frac{\partial P_1}{\partial a_1} & \frac{\partial P_2}{\partial a_1} \\ P_1 & P_2 \end{array} \right]^{-1} \left[ \begin{array}{c} 1MM \\ 0 \end{array} \right] = \left[ \begin{array}{c} 0.1142MM \\ -0.1362MM \end{array} \right]
\tag{28}
$$

Therefore, to carry out the trading is necessary to buy 0.1142MM units of the security with maturity in 2010 for each sale of 0.1362MM units of the security with maturity in 2045.

The absolute financial value of the portfolio is:

$$
abs\,(\phi) \cdot (P_1, P_2) = 203,975.50
$$

After 1 week (5 business days) the operations is undone with the new interest rate curve parameters, price of the securities and yields presenting the following values:

$$
\begin{aligned}
a1 &= (0.085, -0.0012, -0.1326, 0.1826) & (29) \\
(P_1, P_2) &= (0.8776, 0.6831) & (30) \\
(y_1, y_2) &= (0.1034, 0.0880) & (31)
\end{aligned}
$$

To understand what happened to the curve when the coefficient changes from $a_0$ to $a_1$, movement by movement, see Figure (8).

The result forecasted by the model for an increase in level of 100bps is a profit of approximately 10,000, while the effective result (recalculated prices) is a profit of 7,157.16. There are two reasons for the large difference between the two results:

(1) the sale of the level is done in first order – which is not very important, as we will show – and

(2) other risk factors were not considered in the construction of the operation.

To demonstrate that, let us suppose that the coefficients did not change, then the effective result would be a profit of 8,668.71, which is much closer to the approximate profit obtained by *duration*.

### 5.1.2 Duration methodology

If instead of using the Svensson model, the duration was used to operate the level, the resulting portfolio would be:

$$\phi = \left[ \begin{array}{cc} \frac{\partial P_1}{\partial y_1} & \frac{\partial P_2}{\partial y_2} \\ P_1 & P_2 \end{array} \right]^{-1} \left[ \begin{array}{c} 1MM \\ 0 \end{array} \right] = \left[ \begin{array}{c} 0.1260MM \\ -0.1504MM \end{array} \right]$$

With the absolute financial value of the portfolio:

$$abs\left(\phi\right) \cdot (P_1, P_2) = 225,180.50$$

The result predicted by the model for an increase in level of 100 bps is a profit of approximately 11,051.61, while the effective result (recalculated prices) is a profit of 7,901.21. If the coefficients had not changed (only $a_1$), the effective result would have been 9,569.89.

The Svensson model (SV) forecasted and realized results and that of the duration model (D) are shown in Figure 9. The continuous black line shows the results for variations of $\Delta_{a1}$, the red dashed line shows the results for coefficients equal to $a_0$ and variations of $\Delta_{a1}$ and the green dotted line shows the results for coefficients equal to $a_1$ and variations of $\Delta_{a1}$. The graph on the left shows the forecasted and realized results when the operation is done using the Svensson model. The graph on the right shows the same results for duration-based level operation.

As it can be seen in Figure 9, for both methodologies – SV and D –, the forecasted result is more accurate the smaller the shift in $a_1$ (red dashed line). Notice the influence in the result when the coefficients $(a_2, a_3, a_4)$ are different than those in the beginning of the operation (green dotted line). This happens because the portfolio is not immunized against undesirable risks sources, that is, against variations in the factors $(a_2, a_3, a_4)$.. The next example shows how to operate the slope without exposing the portfolio to other risk sources.

Figure 8
Step by step curve movement, from $a_0$ to $a_1$
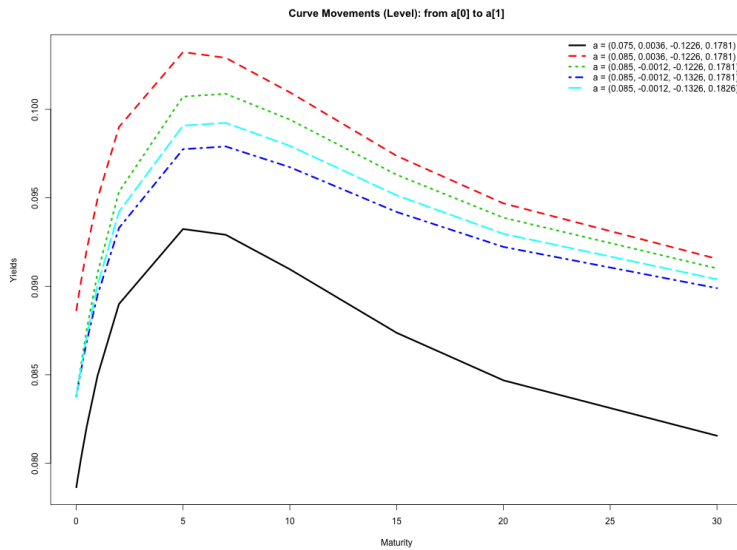


5.2 Example 2: Slope trading neutralizing other sources of risk

Let us suppose that on 07-25-2006 it is expected an increase in the NTN-B curve slope. Suppose that there are only five Brazilian IPCA-indexed bonds in the Brazilian public securities market:

1. NTN-B with maturity 08-15-2008, biannual payment of coupons of 6%pa ($P_1$);

2. NTN-B with maturity 08-15-2010, biannual payment of coupons 6%pa ($P_2$);

3. NTN-B with maturity 05-15-2015, biannual payment of coupons 6%pa ($P_3$);

4. NTN-B with maturity 08-15-2024, biannual payment of coupons 6%pa ($P_4$);

5. NTN-B with maturity 05-15-2045, biannual payment of coupons 6%pa ($P_5$).

The coefficients estimated using $(\lambda_1, \lambda_2) = (0.3, 0.2)$ present the following values:

$$a0 = (a_1, a_2, a_3, a_4) = (0.075, 0.0036, -0.1226, 0.1781)$$

To calculate the sensitivities $\left( \frac{\partial P_i}{\partial a_i}, \frac{\partial P_i}{\partial y_i}, 0.5 \frac{\partial_2 P_i}{\partial y_i^2} \right)$, preços ($P_i$), prices ($P_i$) and durations ($D_i$), suppose that the face value of the securities is 1. Therefore,

$$(P_1, P_2, P_3, P_4, P_5) = (0.9575, 0.8929, 0.8104, 0.7979, 0.7484)$$

Figure 9

Forecasted impacts in the operation procedure "SV" and "D" for different $\Delta_{a1}$ (continuous line) and impacts for different $\Delta_{a1}$ regarding coefficients, except for $a_1$, do not vary (dashed line) and when the coefficients do vary (dotted line)
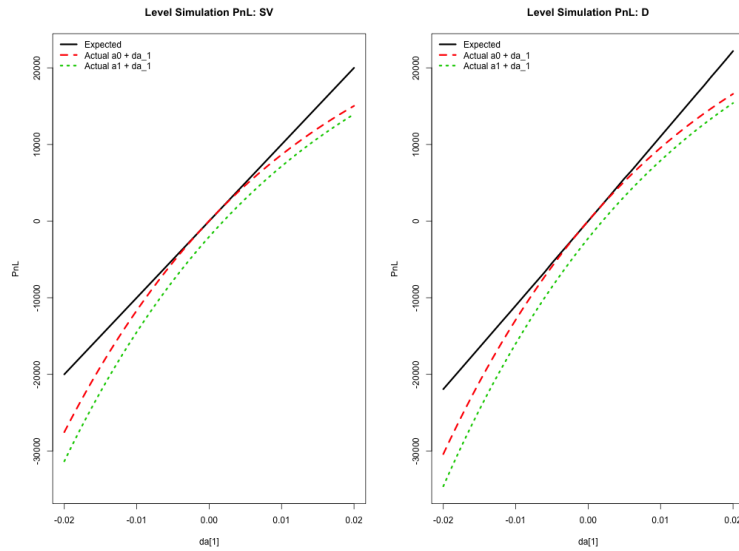
$$(y_1, y_2, y_3, y_4, y_5) \qquad\qquad = (0.0952, 0.0979, 0.0913, 0.0824, 0.0802)$$

$$(D_1, D_2, D_3, D_4, D_5) \qquad\qquad = (1.8942, 3.5067, 6.6127, 9.9611, 12.3885)$$

$$-\left(\frac{\partial P_1}{\partial a_1}, \frac{\partial P_2}{\partial a_1}, \frac{\partial P_3}{\partial a_1}, \frac{\partial P_4}{\partial a_1}, \frac{\partial P_5}{\partial a_1}\right) = (1.8133, 3.1291, 5.3786, 8.1927, 9.9610)$$

$$-\left(\frac{\partial P_1}{\partial a_2}, \frac{\partial P_2}{\partial a_2}, \frac{\partial P_3}{\partial a_2}, \frac{\partial P_4}{\partial a_2}, \frac{\partial P_5}{\partial a_2}\right) = (1.3646, 1.8600, 2.1166, 2.1147, 2.0027)$$

$$-\left(\frac{\partial P_1}{\partial a_3}, \frac{\partial P_2}{\partial a_3}, \frac{\partial P_3}{\partial a_3}, \frac{\partial P_4}{\partial a_3}, \frac{\partial P_5}{\partial a_3}\right) = (0.3669, 0.8602, 1.4992, 1.7116, 1.6152)$$

$$-\left(\frac{\partial P_1}{\partial a_4}, \frac{\partial P_2}{\partial a_4}, \frac{\partial P_3}{\partial a_4}, \frac{\partial P_4}{\partial a_4}, \frac{\partial P_5}{\partial a_4}\right) = (0.2782, 0.7281, 1.5439, 2.0831, 2.0250)$$

$$-\left(\frac{\partial P_1}{\partial y_1}, \frac{\partial P_2}{\partial y_2}, \frac{\partial P_3}{\partial y_3}, \frac{\partial P_4}{\partial y_4}, \frac{\partial P_5}{\partial y_5}\right) = (1.8138, 3.1313, 5.3590, 7.9480, 9.2718)$$

$$0.5\left(\frac{\partial_2 P_1}{\partial y_1^2}, \frac{\partial_2 P_2}{\partial y_2^2}, \frac{\partial_2 P_3}{\partial y_3^2}, \frac{\partial_2 P_4}{\partial y_4^2}, \frac{\partial_2 P_5}{\partial y_5^2}\right) =$$
$$(1.8129, 6.0532, 21.2382, 57.0224, 102.8207)$$

### 5.2.1 Svensson Model

To take advantage of the expected movement the investor should build a directional self-financing position bought on slope in a way that 1 unit of variation in $a_2$ increases the value of the portfolio in 1MM.

To carry out this operation five instruments are necessary: one to leverage the curve's slope, one to build a self-financing portfolio, and the others to immunize the portfolio against variations in the coefficients $(a_1, a_3, a_4)$.

The problem consists in solving the following system of equations:

$$(D_a P)^T \phi \;=\; \begin{bmatrix} 0 \\ 1MM \\ 0 \\ 0 \end{bmatrix} \tag{32}$$

$$\phi \cdot P \;=\; 0$$

The solution is given by:

$$\phi = \begin{bmatrix} (D_a P)^T \\ P^T \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1MM \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.8318MM \\ -9.6520MM \\ 8.6011MM \\ -6.1074MM \\ 2.5313MM \end{bmatrix} \tag{33}$$

Therefore, to carry out the operation it is necessary to buy the vector in quantities

$$(4.8318, -9.6520, 8.6011, -6.1074, 2.5313) \, MM$$

of securities $(1, 2, 3, 4, 5)$.

Given the quantity vector $\phi$, build a portfolio with absolute financing value of:

$$abs\,(\phi) \cdot P = 26,983,773.00$$

and a 5-day carry of $-573.60$.

After one week (five business days) the operation is undone and the new coefficients and prices are:

$$
\begin{aligned}
a1 &= (0.0774, 0.0136, -0.1326, 0.1826) \\
(P_1, P_2, P_3, P_4, P_5) &= (0.9415, 0.8690, 0.7805, 0.7643, 0.7121) \\
(y_1, y_2, y_3, y_4, y_5) &= (0.1052, 0.1063, 0.0972, 0.0870, 0.0845)
\end{aligned}
$$

The result predicted by the model is a profit of $10,000.00$ which is very close to the effective result of a profit of $9,709.59$.

Let us assume that the coefficients did not change, then the effective result is a profit of $9,324.44$. At first it seems odd that the result with varying coefficients is closer to the predicted profit. This happens because in this forecast we do not include the 5-day carry of $-573.60$. When the carry is included in the forecast, the result is a predicted profit of $9,426.30$, much closer to the results without coefficient variation. The difference between the effective results can be explained by the fact that the coefficient $a_1$, representing the movement of level, varies $+24$bps and the portfolio has more short then long positions (negative carry). Because the hedge is made in first order, a large increase in the level coefficient affects the final value of the portfolio, even when the loading of this factor is zero. In this case, the final value of the portfolio will be greater due to an increase in the interest rate curve level.

### 5.2.2 Duration model

**Definition of slope in a Duration model** To trade slope using the Duration model, the investor is faced with a problem: How to define a movement in slope using a duration model? Notice that going from duration to level is relatively easy, but going from duration to slope is a bit more complex. Generally speaking, the models in which the slope is well defined, it is seen as a bear-flattening movement, in which the short rates increase more than the long term ones. Using this fact, the present paper defines slope in two distinct ways:

(1) slope as the inverse of duration $(D \times 1/D)$ and

(2) slope as Svensson loading as a function of duration $(D \times SV\_L2)$.

1. Slope movement as the inverse of duration ($D \times 1/D$): by definition a shock in the slope movement causes the rates to vary in $1/D + 0.1$. For the five securities in this section this means variations of $1/D + 0.1 = 1/(1.8942, 3.5067, 6.6127, 9.9611, 12.2885) + 0.1$. The addition of the 0.1 is necessary to avoid matrix inversion problems in the procedure to build a portfolio that takes advantage of the increase of slope caused by linear dependencies between the matrix lines.

2. Slope movement as Svensson loading in the duration. ($D \times SV\_L2$): by definition a shock in the slope movement causes the rates to vary $\frac{\partial R(\tau)}{\partial a_2}$ in which $\tau = D$ (duration of the security) – see 7.

**Operating the slope in a Duration model** To improve the comparability of the duration with the Svensson model this paper also shows how to trade the slope through duration by neutralizing the level movements ($\frac{\delta P}{\delta y}$) and the convexity movements ($\frac{\delta_2 P}{\delta y^2}$). The paper presents a total of four procedures to trade slope based on duration, as defined below:

1. $D \times 1/D$: Movement of slope as the inverse of duration, without immunization against first and second orders variations in the yield, using securities 1 and 5 and generating 1MM profit with the increase of 1 in the slope.

2. $D \times SV\_L2$: Movement of slope as Svensson loading in the duration, without immunizations against first and second orders variations in the yield, using securities 1 and 5 and generating a profit of 1MM with an increase of 1 in the slope.

3. $D \times SV\_L2 + Dh$: movement of slope as Svensson loading in the duration, with immunization against first order variation in the yield, using securities 1, 3 and 5 and generating a profit of 1MM with an increase of 1 in the slope.

4. $D \times SV\_L2 + Dh + Ch$: Movement of slope as Svensson loading in the duration, with immunization against first and second order variations in the yield, using securities 1, 3, 4 and 5 and generating a profit of 1MM with an increase of 1 in the slope.

$D \times 1/D$: **Movement of slope as the inverse of duration, without immunization against first and second orders variations in the yield** The creation of this operations is based on the following premises:

(1) securities 1 (2008) and 5 (2045) will be used;

(2) the portfolio is self-financing, and

(3) a profit of 1MM with a shock of one unit in the slope.

The result of the procedure is a portfolio with amounts $(\phi_1, \phi_5) = (0.9951, -1.2732)MM$, and an absolute financial value of $1,905,800.00$ and a 5-day carry of $412.82$.

$D \times SV\_L2$: **movement of slope as Svensson loading in the duration, without immunization against first and second order variations in the yield** The creation of this operation is based on the following premises:

(1) the securities used will be 1 (2008), and 5 (2045);

(2) the portfolio is self-financing;

(3) a profit of 1MM with a shock of 1 unit in the slope.

The result of this procedure is a portfolio with amounts $(\phi_1, \phi_5) = (0.5778, -0.7393)MM$, an absolute financial value of $1,106,628.00$ and a 5-day carry of $239.71$.

$D \times SV\_L2 + Dh$: **movement of slope as Svensson loading in the duration, with immunization against first order variation in the yield** The creation of this operation is based on the following premises:

(1) the securities used will be 1 (2008), 3 (2015) and 5 (2045);

(2) the portfolio is self-financing;

(3) a profit of 1MM with a shock of 1 unit in the slope and;

(4) Modified Duration of the portfolio is equal to zero.

The result of this procedure is a portfolio with amounts $(\phi_1, \phi_3, \phi_5) = (0.9311, -1.9990, 0.9732)MM$, an absolute financial value of $3,240,081.00$ and a 5-day carry of $369.16$.

$D \times SV\_L2 + Dh + Ch$: **Movement of slope as Svensson loading in the duration, with immunization against first and second order variations in the yield** The creation of this operation is based on the following premises:

(1) securities 1 (2008), 3 (2005), 4 (2024) and 5 (2045) will be used;

(2) the portfolio is self-financing;

(3) a profit of 1MM with a shock of 1 unit in the slope;

(4) first order variation in the portfolio (*Modified Duration*) equal to zero; and

(5) second order variation in the portfolio (*Convexity*) equal zero.

The result of this procedure is a portfolio with amounts $(\phi_1, \phi_3, \phi_4, \phi_5) = (1.1964, -4.6275, 4.9757, -1.8247)MM$, an absolute financial value of $10,231,736.00$ and a 5-day carry of $305.23$.

The predicted and realized results of the Svenssson model (SV) and of the four duration procedures are shown in pictures 10, 11, and 12. The continuous black line represents the forecast result for variations of $\Delta a_2$, the red dashed line represents the result with the coefficients equal to $a_0$ and variations of $\Delta a_2$ and the green dotted line represents the results with the coefficients equal to $a_1$ and variations $\Delta a_2$.

Notice that for the procedure SV (10), the approximated result of the operation is practically the same as the effective result, regardless of the variation in the coefficients. This shows that, despite the transaction costs, fixating the movements that are not being traded is greatly advantageous, because it increases the precision of the expected result of the operation. Additionally, it shows the power of first order hedging in SV. Despite the hedging being in first order the approximated and the effective results are perfectly adjusted. For procedures $D \times 1/D, D \times SV\_L2, D \times SV\_L2 + Dh$, and $D \times SV\_L2 + Dh + Ch$ (see 10, 11, 12), notice that the approximated and the effective results are perfectly adjusted only for small variations of , and while the other coefficients do not vary (red dashed line).

Figure 10

Forecasted impacts in the procedures SV and D × 1/D for different $\Delta a_2$ (solid line) and impacts for different $\Delta a_2$ when coefficients, except for $\Delta a_2$, do not vary (dashed line) and when the coefficients do vary (dotted line)
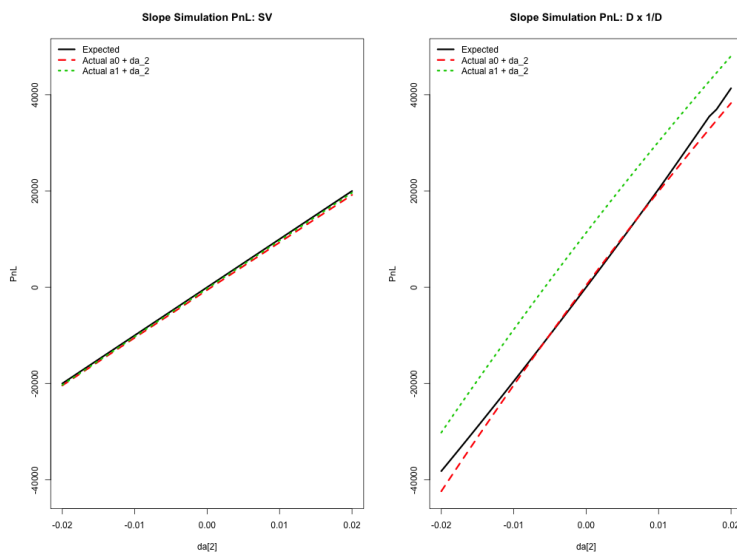


When the other coefficients vary the result is significantly different from the approximate result, regardless of the improvement in slope definition (from $D \times 1/D to D x SV\_L2$), immunization against rates variations in first order (from $D \times$

Figure 11

Forecasted impacts in procedures "D × SV_L2" and "D × SV_L2+Dh" for different $\Delta a_2$ (solid line) and impacts for different $\Delta a_2$ when coefficients, except for $a_2$, do not vary (dashed line) and when coefficients do vary (dotted line)
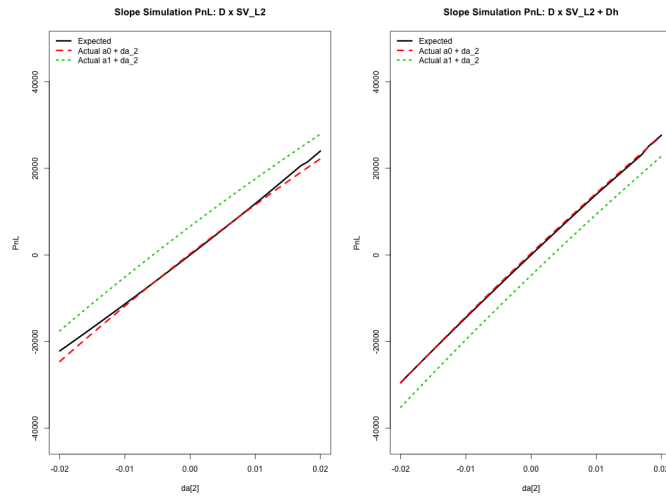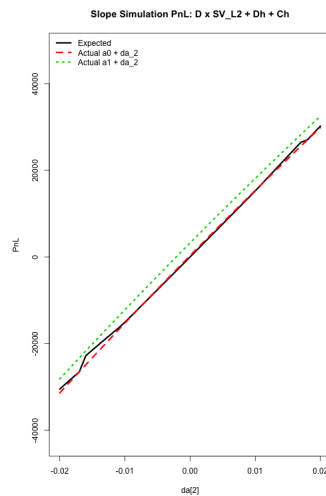


Figure 12

Forecast impacts for procedures "$D \times SV\_L2 + Dh + Ch$" for different $\Delta a_2$ (solid line) and impacts for different $a_2$ when coefficients, except for , do not vary (dashed line) and when coefficients do vary (dotted line)

$SV\_L2toD \times SV\_L2 + Dh$) and immunization against rates variations in first and second orders (from $D \times SV\_L2 + DhtoD \times SV\_L2 + Dh + Ch$). Notice that as the operation procedures via duration become more advanced, the greater the absolute financial value of the portfolio (from approximately 1MM to 10MM). An advantage of the duration model is that, even in the most complex procedure "$D \times SV\_L2 + Dh + Ch$", the absolute financial value is significantly lower than in the Svensson model. Another interesting aspect is that all portfolios created using the duration model have a carry of 5 days positive, while SV does not. Which means that if the market goes sideways (or experiences low volatility) during the operation period, it would be more advantageous to use the duration model instead of the SV model.

The biggest disadvantage of the Svensson model is that, to build the operation, a relative large absolute financial value might be necessary (despite the portfolio being self-financing), since the hedge is the result of a system of equations. On the other hand, for this very reason, the forecasted result is more precise than in procedures which use less securities and a lower absolute financial value.

The criticism to all the procedures described is that to build an operation it is necessary to build a portfolio with short positions for some NTN-B of different the maturities. In practice, a selling position is possible, but the cost of the transaction can be huge.

The criticism of the high absolute financial value and the short position of the portfolio can be easily solved. The first with a credit restriction and the second with a modification in the hedging approach so that it is the solution to an optimization problem with restrictions to short selling. The interesting aspect of this approach is that all securities used to estimate the curve can be used in the construction of the hedge/operation.

## 5.3   Example 3: Factor replicating portfolios

The objective of this example is to show that it is possible to replicate a latent factor using a set of fixed-income securities available in the market. The solution of this problem is important, because it allows us to answer the key question: *what is the price of the latent factor?* Therefore, if it is possible to build a portfolio composed of observable assets in such way that every state of nature of their payments be equal to that of the latent factor, then, under non-arbitrage, the price of both strategies, the portfolio and the latent factor, must be the same.

Let us assume an economy identical to the previous one – same securities, dates, parameters and, therefore, the same curve. Then the coefficients, price of securities and *duration*, are given by the following values:

$$(a_1, a_2, a_3, a_4) = (0.0733, 0.0014, 0.1896, -0.1252)$$

$$
\begin{aligned}
(P_1, P_2, P_3, P_4, P_5) &= (0.9603, 0.8952, 0.8130, 0.8048, 0.7589) \\
-\left(\frac{\partial P_1}{\partial a_1}, \frac{\partial P_2}{\partial a_1}, \frac{\partial P_3}{\partial a_1}, \frac{\partial P_4}{\partial a_1}, \frac{\partial P_5}{\partial a_1}\right) &= (1.8226, 3.1412, 5.4012, 8.3013, 10.2448) \\
-\left(\frac{\partial P_1}{\partial a_2}, \frac{\partial P_2}{\partial a_2}, \frac{\partial P_3}{\partial a_2}, \frac{\partial P_4}{\partial a_2}, \frac{\partial P_5}{\partial a_2}\right) &= (1.3709, 1.8664, 2.1247, 2.1371, 2.0373) \\
-\left(\frac{\partial P_1}{\partial a_3}, \frac{\partial P_2}{\partial a_3}, \frac{\partial P_3}{\partial a_3}, \frac{\partial P_4}{\partial a_3}, \frac{\partial P_5}{\partial a_3}\right) &= (0.3693, 0.8638, 1.5053, 1.7323, 1.6483) \\
-\left(\frac{\partial P_1}{\partial a_4}, \frac{\partial P_2}{\partial a_4}, \frac{\partial P_3}{\partial a_4}, \frac{\partial P_4}{\partial a_4}, \frac{\partial P_5}{\partial a_4}\right) &= (0.2800, 0.7313, 1.5505, 2.1100, 0.0718)
\end{aligned}
$$

The solution to the problem is to build portfolios $i = 1, 2, 3, 4$; in such way that one unit of variation in $a_1$ increases the value of the portfolio $i$ in 1 financial unit. Because just four instruments are enough to carry out this operation, with no loss of generality, remove the first security – the one with the shorter maturity (2008). To build the $i$ portfolios that replicate the $i$ factors it is necessary to solve the linear system:

$$
(D_a P)^T \phi = I_4 \tag{34}
$$

onde $D_a P$ is the matrix of derivatives of the prices of the four chosen securities in relation to the factors;

$I_4$ is the identity matrix (4×4)

The solution is given by:

$$
\phi = \left[(D_a P)^T\right]^{-1} I_4 = \begin{bmatrix}
0.0721 & -2.1634 & 5.3773 & -2.5073 \\
-0.2077 & 2.7196 & -11.4152 & 7.4346 \\
0.6736 & -2.4079 & 10.9315 & -9.6599 \\
-0.5560 & 1.1806 & -4.4882 & 4.6765
\end{bmatrix} \tag{35}
$$

Each column $i$ represents the amount of the 4 securities that an agent must buy to replicate the $i$-th factor.

The price of the factor $i$ is given by the linear combination of the prices of the securities weighted by the values of the $i$-th column of the matrix $\phi$. Or, in a more general description, given by:
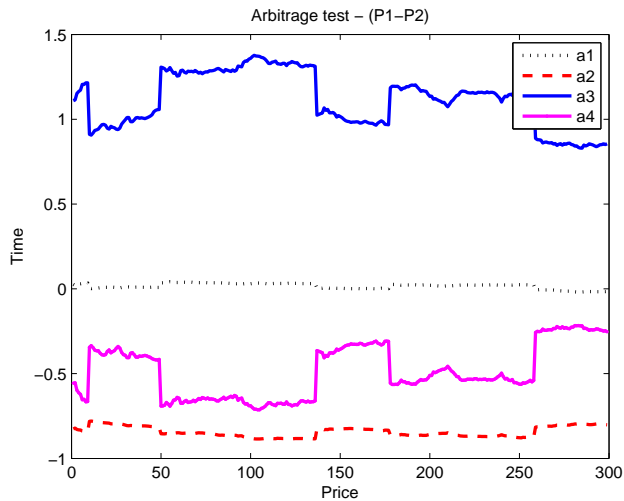
$$
P_{factors} = P^T \times \phi \tag{36}
$$

In the problem above, $P_{factors}$ is equal to the vector:

$$
P^1_{factors} = (0.0158, -0.7676, 0.9248, -0.4255)
$$

where each entry $i$ of the vector represents the price of the $i$-th vector.

Figure 13
Evolution in the difference in price of the portfolios



It is worth mentioning that if a different group of securities is used, the price vector will change. The price vector of the factors built by a different group of securities is:

$$P^2_{factors} = (0.9244, -2.9127, 11.4898, -10.9082)$$

This large difference exists because we do not use the whole yield curve to build the replicated factors. That is, instead of using all securities available in the market, we have used only four securities of the curve.

At first, when buying the first portfolio that replicates factor 1 and selling the second portfolio that also replicates factor 1 there is a profit of $(0.9244 - 0.0158)$ and the investor is free from the risk in infinitesimal movements in all factors, including the first. Unfortunately this is not an arbitrage opportunity, because in the future it will be necessary to sell this portfolio, and when doing so all the profit from the beginning of the operation will, in average, be lost in the end.

To illustrate this last point, one of the factors was replicated with two portfolios of four securities each – for each one of the factors the same portfolio was used – to obtain two price vectors ($P^1$ and $P^2$) – one for each factor. Figure 13 shows the evolution of the difference of the price vector of these portfolios ($P^1 - P^2$) without rebalancing in each period.

Despite the fact that the prices of the latent factors using the observable assets are dependent on the portfolio, the behavior of the price is very similar for both portfolios. Figures 14, 15, 16 and 17 show the evolution of prices of factors $i =$

$1, 2, 3, 4$; respectively.

Observe that the prices of the implicit factors in portfolio 2 always have a larger absolute value. This is due to the fact that their *duration* is longer than portfolio 1. Another relevant aspect is the spikes in the price of the factors. Bear in mind that the securities pay coupons on different dates. Therefore, the spikes are the result of coupons being paid by only a part of the securities in the factor replicating portfolio.

Figure 14
Factor $a_1$ price evolution
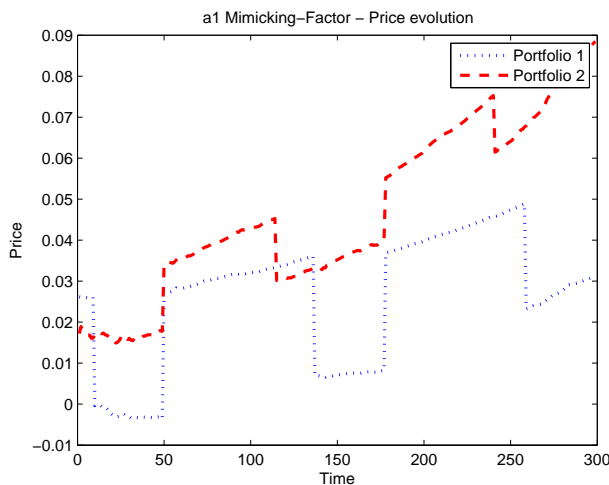


The fact that the price of some factors is negative means that the agents are averse to these negative price factors. In other words, they expect the behavior of the factors to be the opposite. Notice that the price of the second curvature is negative, while in the second one it is positive, that is, the curvature movements cancel each other out. On the other hand, the fact that the slope has negative price throughout the sample suggests that the agents expected an increase in the slope of the curve. This is consistent with the empirical evidence of the decrease in the real interest rate in the Brazilian economy.

Additionally, a decrease in the price of the slope throughout the period is observed (that is, it becomes more negative), indicating that the agents were reevaluating the probability of the slope increasing and, ultimately, inverting (because, in the beginning of the exercise the curve is decreasing and convex).
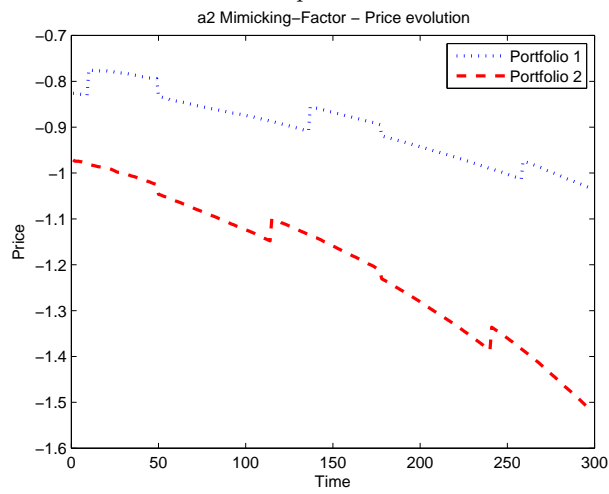
Figure 15
Factor $a_2$ price evolution



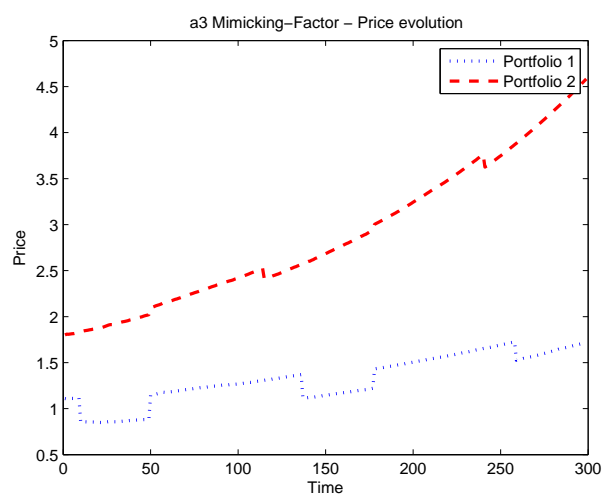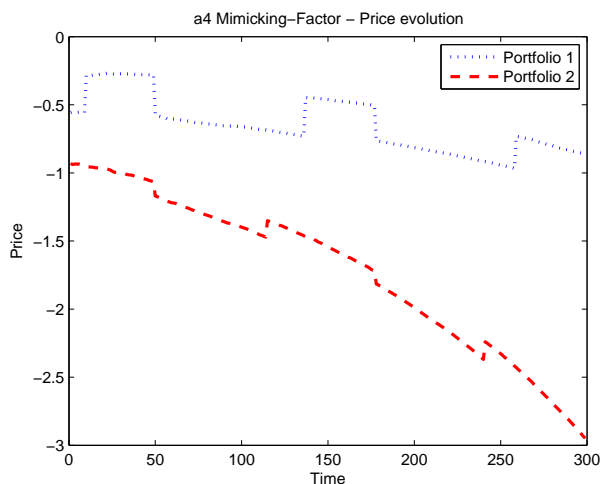Figure 16
Factor $a_3$ price evolution

Figure 17
Factor $a_4$ price evolution



## 6. 6 Extraction of the implicit expectations

This section presents the methodology for the construction of inflation curves (IPCA) implicit in the derivatives and public bonds markets. After its construction, a comparison between the expectations produced in both markets is carried out.

Let $R_t^{nom}(\tau)$ be the nominal interest rate and $R_t^{real}(\tau)$ the real interest rate. According to Fisher rule, the inflation rate is the nominal interest rate minus the real interest rate:

$$R_t^{\inf}(\tau) = R_t^{nom}(\tau) - R_t^{real}(\tau)$$

where $\tau$ is the maturity and t represents the instant in time.

In the Brazilian case, the inflation curve can be built in two ways: from a group of bonds created by the fixed bonds (LTNs and NTN-Fs), and from inflation-indexed bonds (NTN-Bs or NTN-Cs) and from the group of two types of derivatives, DI-Futurp and IPCA coupon (SDL) or IGPM coupon (SDM). It is possible to decompose a curve, be it the inflation or the interest rate curve, into two components: expectation of future rates and risk premium. The first explains the largest part of variations in the short end of the term structure, while the second explains the largest part of variations in the long part of the curve. Therefore, for shorter maturities, the difference between the curves, nominal and real, seems a good measure of inflation expectation, since the interference of the risk premium is small.

The next step is choosing the method to estimate the real and nominal interest

rate curves. These depend only on the group of observable variables, as discussed in the estimation section. For example, the NTN-Bs do not have rates that are directly observable, only yield to maturities and prices. In this case, the method chosen must be price based.

Another concern regarding the estimation procedure is knowing whether the model proposed for the construction of the curve adjusts well to the data. In this case the focus should be the boundaries of the curve: the short and long end. The first is, in general, more problematic.

To insure good data adjustment of the model, especially for short maturities, we allow an increase in the group of securities to include instruments with maturities of one day. For the nominal curves we used the Selic overnight rate or the interbank overnight rate (DI), and for the real curves we used the difference between the nominal one day rate and the projected inflation released by Anbima.[14]

## 6.1  Estimation of the curves using the grid methodology

To estimate the curves we used a grid for the parameters of scale (as described in the estimation section). Each parameter of scale can assume 31 values,[15] and for each pair of values, the model of the term structure was estimated for the four curves in each period of time. For the two term structures of the securities market (real and nominal) the price loss function (11) was used, because the rates are unobservable. For the derivatives market term structure, the rate loss function (13) was used.

Observe that the rate loss function, for a given pair of parameters of scale in the grid, is linear in the parameters, which makes it possible to use linear techniques to find its solution. The price loss function, on the other hand, is completely non-linear in the parameters and must be solved using non-linear procedures to find the factors. Both optimizations were solved with a condition of positivity for the rate level factor.

Once the factors were optimized, the criteria for choosing the pair of scale parameters and optimum factors were the average of absolute residuals.

For the derivatives market, the result was good, but for the securities market a problem was observed: for some dates, the difference between the real one-day rates, estimated and observed, was as high as 1000bps. To avoid this problem we suggest a transformation in the loss function in terms of price [equation (11)]. Below we have the new loss function used in the estimation of the real and nominal

---

[14]This projection was chosen because its calculations uses the Updated Nominal Value (UNV) of the Brazilian inflation-indexed public debt bonds.

[15][(0.1 : 0.1 : 1), (1.2 : 0.2 : 2), (2.25 : 0.25 : 4), (4.5 : 0.5 : 8)], where the middle value in each parenthesis denotes the space skipped between the values on both extremes.

interest rate curve for the securities market.

$$\min_{a_{t,1}, a_{t,2}, a_{t,3}, a_{t,4}} \sum_{j=1}^{J} w_{t,j} \left( -\log\left( \overline{P_{t,j}} \right) + \log\left( P_{t,j} \right) \right)^2 \tag{37}$$

$w_{t,j} = \left( 1/T_{t,j} \right)^2$

$P_{t,j}$ is the price of the security $j$ given by equation (8) of the model in the instant in time $t$';

$\overline{P_{t,j}}$ is the price observed in the market of the security $j$ in the instant in time $t$;

$\omega_{t,j}$ is the weight which attributes the degree of importance of the security $j$ in the instant in time $t$;

$T_{t,j}$ is the maturity of the security $j$ in the instant in time $t$.

The idea of the transformation is to change the focus of the problem from the prices to the rates, and with that, guarantee a small error for the rates of all the securities with only one payment left. Keep in mind that the price of a security with only one payment left is given by $P_{t,j} = exp\left( -y_{t,j} T_{t,j} \right)$.

Another possible weight would be the duration $\left( w_{t,j} = \left( 1/D_{t,j} \right)^2 \right)$. In our experiment, the choice of the duration or maturity as a weight is not very relevant, because we are interested in finding the expectation of inflation in a period of, at most, 3 years. It is important to mention that this transformation can only be made for the construction of the NTN-B curve, because this is the only one in which the problem was observed.

Observe in Figure (18) the large difference in the beginning of the real interest rate curves in the securities market, when they are built using the old and the new method. As explained in the estimation section, the loss function based in prices can present a lot of errors for short maturities (the error for maturities of one day is approximately 1000 bps). The new function proposed solves this problem. See the graph on top of Figure (18).

Observe in Figure (19) that the average curves of inflation are very different in the short part: maturities up to a year, but practically the same for maturities greater than a year.

This fact can be more clearly seen in table 20, which presents the average and standard deviation of the nominal, real and inflation rates estimated for the securities market (Selic, Real_Selic and Inflation_Selic) and for the derivatives market (DI, Real_DI, and Inflation_DI). Observe that the average breakeven inflation implicit in the derivatives market has a higher level for all the selected maturities then the ones implicit in the securities market. This difference begins in 100 bps for maturities of 21 business days and gradually decreases to 13 bps for maturities of 756 business days. On the other hand, the standard deviation for the inflation measure based on derivatives is much bigger than that obtained with the public securities, especially for short maturities.

This difference in the measures of inflation can be almost entirely explained by movements of the real rates in both markets, since both nominal interest rate curves have averages that are practically the same, as well as very similar standard deviations. The high liquidity of both type of instruments used to build the nominal curve prevents greater distortions. Curves with nearly identical characteristics must have been practically identical in shape, otherwise arbitrage would be possible.

The difference in the breakeven inflation is counterintuitive. There can be two explanations for this fact:

(1) there is an opportunity for arbitrage that is not taken by the market agents, or

(2) there is a microstructure that can explain this discrepancy.

The IPCA [SDL] coupon market has low liquidity when compared with the volume of operations with inflation-indexed securities. Then it is possible that this difference be explained by the demand of a larger premium by the agents to compensate the lower liquidity of that instrument. Future research could answer the question: "Does the microstructure explain the difference or is there an unseen opportunity for arbitrage?"

Figure 18
Curves when the loss function is based in a pricing error vs. Curves when the loss function is based on rate error
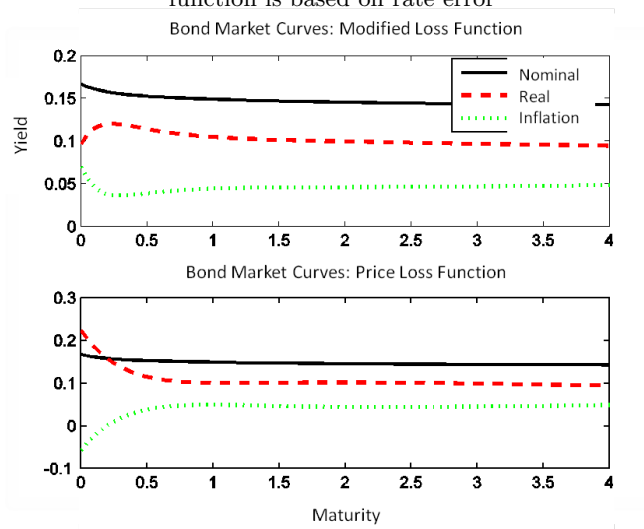
Figure 19
Average of the estimated curves using derivatives (top), using public bonds (middle)
and difference of the inflation curves obtained using bonds and derivatives, respectively,
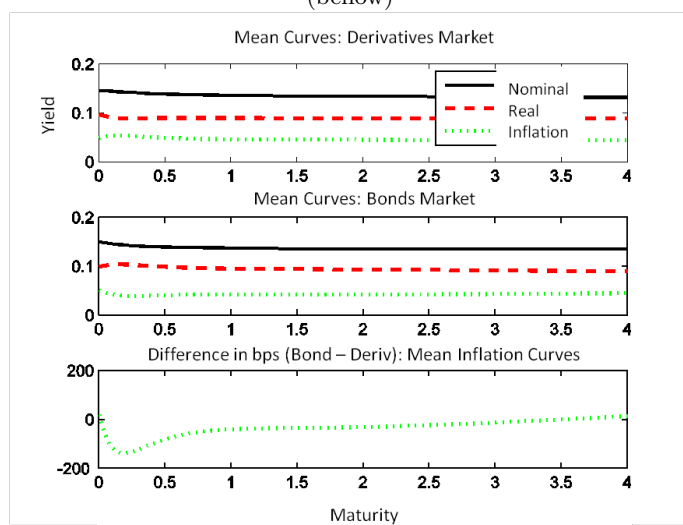(bellow)



Figure 20 – Table comparing the average and standard deviation of the inflation
expectations (annually) throughout time for selected future durations

| Duration (bus days) | DI | Selic | Real_DI | Real_Selic | Inflation_DI | Inflation_Selic |
|---|---|---|---|---|---|---|
| 21 | 0,1445 | 0,1461 | 0,0911 | 0,103 | 0,0534 | 0,0431 |
|  | 0,0147 | 0,0155 | 0,0142 | 0,0129 | 0,0224 | 0,0093 |
| 42 | 0,143 | 0,1436 | 0,0892 | 0,1037 | 0,0538 | 0,04 |
|  | 0,0143 | 0,0143 | 0,0142 | 0,0147 | 0,0217 | 0,0052 |
| 63 | 0,1415 | 0,142 | 0,0889 | 0,1027 | 0,0526 | 0,0392 |
|  | 0,0138 | 0,0138 | 0,0117 | 0,0153 | 0,0181 | 0,0045 |
| 126 | 0,1385 | 0,1393 | 0,0897 | 0,0987 | 0,0488 | 0,0406 |
|  | 0,013 | 0,0132 | 0,0085 | 0,0132 | 0,0099 | 0,0035 |
| 189 | 0,1368 | 0,1379 | 0,0898 | 0,0962 | 0,047 | 0,0418 |
|  | 0,0128 | 0,0132 | 0,0093 | 0,0111 | 0,0076 | 0,0034 |
| 252 | 0,1358 | 0,1371 | 0,0896 | 0,0949 | 0,0462 | 0,0421 |
|  | 0,0127 | 0,0132 | 0,0097 | 0,0101 | 0,0069 | 0,0037 |
| 378 | 0,1346 | 0,1358 | 0,089 | 0,0939 | 0,0455 | 0,042 |
|  | 0,0126 | 0,0131 | 0,0097 | 0,0096 | 0,0062 | 0,0037 |
| 504 | 0,1337 | 0,1351 | 0,0886 | 0,0932 | 0,0451 | 0,0419 |
|  | 0,0125 | 0,013 | 0,0096 | 0,0097 | 0,0059 | 0,0035 |
| 756 | 0,1325 | 0,1344 | 0,0882 | 0,0915 | 0,0442 | 0,0429 |
|  | 0,0124 | 0,013 | 0,0095 | 0,0099 | 0,0056 | 0,0034 |

## 7. Conclusions

The present paper used a statistical parametric model to build the interest rate curves when the interest rates are non-observable. Additionally, it discussed and demonstrated, using Brazilian inflation-indexed public debt securities, how it is possible to perform a hedging of fixed income portfolios and to operate curve movements (level, slope and curvature) based on a parametric model.

The paper presented a hedging approach that is an alternative to the one suggested by Litterman and Scheinkman (1991) to build immunized portfolios when the interest rates are unobservable, as it is the case of most international and domestic debt markets in many emerging market economies.

The approach is interesting, because it starts with a parametric model, in which the risk factors are represented by their parameters, with well-defined interpretations: level, slope and curvature. And demonstrates that the immunization or change in exposure in relation to the parameters is equivalent to the portfolio immunization using principal components suggested by Litterman and Scheinkman (1991), because the factors of the model generate the same orthogonal factors subspace resulting from the analysis of principal components directly on the rates (see Almeida et al., 2003).

Simulations of the hedging strategy and or the change in exposure in relation to the parameters for the Brazilian inflation-indexed public debt securities show that, despite being done in first order ($duration), the operation is efficient. Because, since the factors have interpretations of level, slope and curvature, the first order operation in relation to the coefficients is approximately equivalent when considering greater orders – such as convexity – instead of just the duration. Even when compared with duration and convexity based procedures, the proposed methodology is more efficient, especially when the curve presents movements that are different from the ones used in the curve operation. It is also important to point out that the paper proposes two distinct ways of defining the slope movements under the duration and convexity model which had never been described before in the fixed-income literature.

The paper also demonstrated how to build and price portfolios that replicate the risk factors (non-negotiable assets) starting from existing assets in the economy. From their prices it was possible to extract information about the market agents on future movements of the interest rate curves: the negative price of the slope indicates that the agents value an increase in the slope of the curve, which is consistent with the empirical evidence of a decrease in the real interest rate in Brazil. On the other hand, the decrease in price of the slope throughout time suggests that the agents were reevaluating the probability that the slope would increase, and ultimately, invert (because in the beginning of the exercise the curve is decreasing and convex).

Natural extensions would be trying to explain the latent factors using economic fundamentals, that is

(1) with macroeconomic fundamentals – inflation, production, consumption and investment – and/or

(2) with microeconomic fundaments, such as agents preference and production functions, which would allow us to give deeper economic interpretations.

To this end, authors such as Ang et al. (2007), Bernanke et al. (2004), Gallmeyer et al. (2007), BeKaert et al. (2005), to name a few, have recently tried to describe this relation, although in the framework of dynamic term structure models.

Other extensions would be: empirically testing the robustness of the immunization strategies in and out of sample, using credit restrictions and second order optimal conditions, that is, to use the convexity of the portfolio to the maximum to carry out a hedging operation that surpasses the limit of portfolio protection. Additionally, test the hedge model by solving the optimizations instead of a simple equation system. This last procedure could also be used to obtain the approximated price of the factors in a parametric model using all the securities in the market, instead of just four.

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