Another Proof of the Existence of GEI Equilibrium with Default and Exogenous Collateral*

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Abstract

The existence of equilibria in a GEI model with default where short sales are backed by collateral (henceforth: exogenous collateral economy), has been demonstrated by Geanakoplos and Zame in a 2002 version of their paper ("Collateral, Default and Market Crashes"). On the other hand, Araújo et al. (2005) have recently provided a characterization, via non arbitrage, of prices of defaultable assets backed by collateral. The aim of this paper is to provide an alternative proof of the existence of equilibria in an exogenous collateral economy by combining the demand approach used by Geanakoplos and Zame, and the characterization of collateralized asset prices offered by Araújo, Fajardo and Pascoa.

Keywords: Incomplete Markets, Exogenous Collateral, Arbitrage Opportunity.

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1. Introduction

Since Arrow and Debreu introduced their general equilibrium model with complete markets in 1954, and after Radner generalized it in 1972, default has been prohibited by assumption. However, although default seems to be a sign of disequilibrium, the pioneering works of Dubey et al. (1990), Dubey et al. (1995), Geanakoplos and Zame (1997) and their more recent versions, (2004) and (2002) respectively, showed the contrary. The former uses utility penalties to discourage default while the latter requires collateral from the borrowers for each asset sold. Lenders will seize collateral in case of default. Without a doubt the collateralized assets represent a considerable amount of money in modern economies. Similar to Geanakoplos and Zame (2002), this work is concerned with exogenous collateral as a means of enforcing promises. For ample discussion on the role of collateral in securing loans we refer to Geanakoplos (1996).

The purpose of this paper is to show that the standard conditions on fundamentals of the economy (including utilities, endowments, collateral requirements, asset returns, and depreciation structure) are sufficient to guarantee the existence of an equilibrium in an exogenous collateral economy. Such a result is in sharp contrast to the well-known Hart's example of nonexistence of an equilibrium in frictionless incomplete markets with no collateral requirement or default. This result has recently been proven by Genakoplos and Zame in a 2002 version of their paper ("Collateral, Default and Market Crashes"). These authors use the demand approach without mentioning any characterization of asset prices.

On the other hand, the prices of collateralized assets have recently been characterized via arbitrage by Araújo et al. (2005) for the case in which uncertainty is represented by a finite number of states of nature. For the case of a continuum of states this characterization has recently been extended by Orrillo (2002). In contrast to the methodology used by Geanakoplos and Zame (2002), Araújo et al. (2005) used their asset price characterization to establish their existence result without using the demand approach. Instead, they used the generalized game approach of Debreu (1952). More recently, the result of Dubey et al. (1995) has been extended (using the methodology of Debreu again) to the case of a multi-period model by Araújo et al. (2002).

The fundamental result of this paper is the demonstration of equilibria for an exogenous collateral economy. To reach our goal, we combine the characterization of asset prices offered by Araújo et al. (2005) and the demand approach used by Geanakoplos and Zame (2002). The proof follows lines similar to the usual proof of existence in an exchange economy with complete market structure (see Hildenbrand (1974), or the proof followed by Werner (1985) or Zhang (1996) in the case of incomplete financial markets). The most outstanding feature of our proof is that, even in the presence of real assets,[1] the demand function, having arbitrage-free prices as its domain, satisfies the boundary condition.

1.1 Related literature

The proof of the existence of equilibrium with real assets[2] is not a problem if one has lower bounds on short sales (see Radner (1972)). However, without such bounds, Hart (1975) has pointed out that equilibrium may fail to exist. This is due to the lack of continuity of budget correspondence, which in turn leads to a discontinuity in the consumers' demand correspondences, and hence the usual techniques cannot be applied to prove the existence of equilibrium. There are some answers to Hart's problem in the literature. The first is that, in general, Hart's example of nonexistence of an equilibrium is rare. As was shown for the complete market case by Repullo (1986) and Magill and Shafer (1990) – in the incomplete market case by Duffie and Sahafer (1985, 1986), Hirsch et al. (1990) – Radner equilibria exist generically. A second answer, proposed initially by Arrow (1953), but more forcefully put forward by Geanakoplos and Polemarchakis (1986), is to express all payoffs in terms of a numeraire good. A third answer, pursued by Cass (1984), Werner (1985), Duffie (1987) and Zhang (1996) is to propose restrictions on asset returns, such as having nominal returns.

This paper is organized as follows: in section 2, we describe characteristics of the exogenous collateral model, establish assumptions of the model, and end by stating the notion of equilibrium. In section 3, we define demand correspondences, and prove that they are well-defined on the set of arbitrage-free prices. In section 4, we state and prove our main result. We offer some concluding remarks in section 5 and end with an appendix containing the proofs of lemmas.

Miscellaneous Notation

By abuse of notation, we always use the same letter for both set and its cardinality. Thus, we will always write: $A = \{1, \ldots, A\}$. For $C = (C_1, \ldots, C_J) \in R^{JL}, C_j \in R^L; \varphi \in R^J$; and $Y \in R^{L\times L}$ we set: $C\varphi = \sum_{j\in J} C_j\varphi_j \in R^L, YC = \sum_{j\in J} YC_j \in R^L$. For any two vectors $x, y \in R^m, xy = \sum_{i=1}^m x_i y_i \in R$ is the usual scalar product. Therefore, $\forall p \in R^L, pC\varphi = \sum_{j\in J} pC_j\varphi_j$.

[1] The excess demand correspondence does not need to satisfy the boundary condition.

[2] Assets that promise to deliver a bundle of physical goods.

The set $\triangle_+^m := \{x \in R_+^m : \sum_{i=1}^m x_i = 1\}$ is the positive simplex of R^m. The map $\Pi_{i,j} : R^{m_1} \times \cdots \times R^{m_k} \to R^{m_i} \times R^{m_j}$ is the linear projection defined as $\Pi_{i,j}x = (x_i, x_j) \in R_i^m \times R_j^m, x = (x_1, \ldots, x_k) \in R^{m_1} \times \cdots \times R^{m_k}$. If $M \in R^{m \times n}$ is a matrix of order $m \times n$, and $\beta \in R^m, y \in R^n$ are any two vectors, in the product $\beta My \in R$, the vector β will be a line vector and y a column vector. Finally, $[a-b]^+ = \max\{a-b, 0\}$ and $\mathbf{1}$ will denote an $n-$ dimensional vector which has all its coordinates equal to 1.

2. The Model

We consider a two-period exchange economy with a finite number H of consumers and L commodities. There are S states of nature to be revealed in the second period, and in the first period there is just one state of nature (called state 0), in which H agents trade in L commodities and J assets. Short sales are backed by collateral and default is permitted. The collateral is modeled by the physical commodity bundle $C = (C_1, \ldots, C_J)$ where the $j-th$ vector backs the sale of one unit of asset $j \in J$. We assume that the collateral is depreciated according to a positive linear transformation, $Y_s : R^L \to R_+^L$, which depends on each state of nature $s \in S$ to be revealed in the second period. Let us denote the matrix that represents said positive linear transformation by $Y_s = [Y_s^1 \ldots Y_s^l \ldots Y_s^L] \in R_+^{L \times L}$ where $Y_s^l \in R_+^L, l \in L$ is its $l-$ th column.

Each asset $j \in J$ is characterized by its promise $A^j \in R_+^{S \times L}$ and a bundle of goods $C_j \in R_+^L$. Agents are characterized by their utility function $U^h : R_+^{L(S+1)} \to R$ and their initial endowments $\omega^h \in R_+^{L(S+1)}$. As default is permitted, each agent has the option of delivering less than he promised. If we assume that the commodity price system is $p \in R_+^{L(S+1)}$, the value of the promise, in each state s, is $P_s A_s^j$. Let D_s^{hj} denote what agent h decides to deliver. As the only consequence of default is the seizure of the collateral, it then follows that any rational borrower will choose to deliver the minimum of the face value and the depreciated collateral value. In other words, default is strategic. Similarly, each lender expects to receive only the minimum between the claim and the market value of the depreciated collateral. Thus, the delivery on asset j in the second period is defined to be

$$D_s^j = \min\{p_s A_s^j, p_s Y_s C_j\}$$

Formally, we define our economy as follows:

Definition 1 *An exogenous collateral economy is a collection*

$$\mathcal{E} = (\mathcal{U}, \omega, (\mathcal{A}, \mathcal{C}), \mathcal{Y})$$

where $U = (U^1, \ldots, U^H)$ and $\omega = (\omega^1, \ldots, \omega^H)$ are the profiles of utility functions and initial endowments of the H agents. The pair $(A, C) = (A^j, C_j)_{j \in J}$ denotes

the J assets consisting of promises and the collateral protecting the sale of assets, and finally Y is the depreciation structure to which both consumption goods and those serving as collateral are subjected. A price system for this economy is a vector $\pi \in R_+^J$ of securities, and a state-contingent consumption price vector $p \in R_+^{L(S+1)}$.

Definition 2 *Given a price system $(p, \pi) \in R_+^{L(S+1)} \times R_+^J$ a consumption-portfolio choice $(x, \theta, \varphi) \in R_+^{L(S+1)} \times R_+^{2J}$ is budget-feasible for agent $h \in H$ if*

$$p_o(x_o + C\varphi - \omega_o^h) + \pi\theta \leq \pi\varphi \tag{1}$$

$$p_s(x_s - \omega_s^h - Y_s x_o) \leq D_s\theta + (p_s Y_s C - D_s)\varphi, \quad s \in S \tag{2}$$

The budget constraint (1) states that the cost of a net purchase of goods $p_o(x_o + C\varphi - \omega_o^h)$ plus the lending $\pi\theta$ (due to the purchase of assets) cannot exceed the borrowing $\pi\varphi$ (due to the sale of assets). The budget constraint (2) tells us that after $s \in S$ is realized at date 1, the consumer must again decide on his net purchases of goods $p_s(x_s - \omega_s^h - Y_s x_o)$, which must be financed from receipts of assets that he purchased in the first period, and from net deliveries the agent makes on assets he sold.

2.1 Equilibrium and assumptions

Consumption-portfolio choices satisfying the budget constraint (1) and (2) define consumer h' budget set:

$$B^h(p, \pi) = \{(x, \theta, \varphi) \in R_+^{L(S+1)} \times R_+^{2J} : (1) \text{ and } (2) \text{ are satisfied}\}$$

Definition 3 *Given (p, π), a budget-feasible choice (x, θ, φ) for agent h is optimal, if there is no budget-feasible choice (x', θ', φ') for agent h such that*

$$U^h(x'_o + C\varphi', x'_{-o}) > U^h(x_o + C\varphi, x_{-o})$$

The demand correspondence Ψ^h for agent h is defined by:

$$\Psi^h(p, \pi) = \{(x, \theta, \varphi) \in B^h(p, \pi) : (x, \theta, \varphi) \text{ is optimal for agent } h\}$$

Definition 4 (Exogenous Collateral Equilibrium) *An exogenous collateral equilibrium for the economy $\mathcal{E}$ is a price system (p, π), and a collection $(x^h, \theta^h, \varphi^h)_{h \in H}$ of optimal choices for the respective agents. In addition, given (p, π), all markets clear:*

$$\sum_{h \in H}(x_o^h + C\varphi) = \sum_{h \in H} \omega_o^h, \sum_{h \in H} x_s^h = \sum_{h \in H}[\omega_s^h + Y_s(x_o + C\varphi^h)], \forall \in S$$

$$\sum_{h \in H} \theta^h = \sum_{h \in H} \varphi^h.$$

The question is now to know whether there is a price system for which our economy $\mathcal{E}$ has an equilibrium. To obtain this we need to make some assumptions about the characteristics of our economy.

2.2 Assumptions

(1) Each agent $h \in H$ is assumed to have continuous, concave, and strictly increasing utility function $U^h : R_+^{L(S+1)} \to R$;

(2) $\omega_s^h >> 0, \forall (h,s) \in H \times S^*$, being $S^* = S \cup \{0\}$;

(3) $C_j \neq 0$, $\forall j \in J$;

(4) $Y_s^l \neq 0, \forall (s,l) \in S \times L$;

(5) $A_s^j \neq 0, \forall (j,s) \in J \times S$.

Remarks (1) and (2) are standard assumptions, so they do not deserve any commentary. As the collateral is the only method of enforcing promises, (3) is quite natural: any asset which demands no collateral requirement would deliver nothing, and therefore will have zero price in equilibrium.

3. Arbitrage and Individual Demand Correspondence

In this section, we assume that there is no arbitrage opportunity in the financial markets where the default is allowed and short sales are backed by exogenous collateral. Araújo et al. (2005) have characterized, via non-arbitrage, the collateralized asset prices. Although their result was obtained in the endogenous collateral setting, the authors provided a similar result for the exogenous case (see Corollary 1 below) which will be fundamental for our purpose.

For completeness, we state the characterization of the asset price obtained by Araújo et al. (2005) with both endogenous and exogenous collateral. For this work only the latter will be necessary.

Theorem 1 (Araújo - Fajardo - Pascoa) *There are no arbitrage opportunities if and only if there exists $\beta \in R_{++}^S$ such that for each $j \in J$*

$$\pi_2 < \sum_{s \in S} \beta_s D_s^j + (p_o M_j - \sum_{s \in S} \beta_s p_s Y_s M_j)$$

$$\pi_1 \geq \sum_{s \in S} \beta_s p_s N_s, \text{ and } p_o > \sum_{s \in S} \beta_s p_s Y_s$$

π_2 is the market price of collateralized asset, D_s^j is the delivery made by the borrowers who constitute endogenous collateral M. π_1, and N_s are the derivative prices and returns, respectively.

As already stated, when the collateral is exogenous these same authors obtained the following important corollary.

Corollary 1 *There are no arbitrage opportunities if and only if* $\exists \beta \in R^S_{++}$ *such that*

$$\sum_{s\in S} \beta_s D^j_s \leq \pi_j < \sum_{s\in S} \beta_s D^j_s + (p_o C_j - \sum_{s\in S} \beta_s p_s Y_s C_j) \tag{3}$$

which implies

$$p_o C_j - \sum_{s\in S} \beta_s p_s Y_s C_j > 0, \text{ and } p_o C_j - \pi_j > 0, \forall j \in J$$

Remarks

- Since utility functions are assumed to be strictly increasing, commodity prices equal to zero will be ruled out by assumption;
- Let $\mathcal{A}$ be the set of all commodity - asset prices $(p, \pi) = (p_o, p_1, \ldots, p_S, \pi) \in R^L_{++} \times R^{LS}_{++} \times R^J$ that admit no arbitrage opportunity. That is, those that satisfy Corollary 1. Denote the following set by $\mathcal{N}$

  $$\Pi_{1,S+1}\mathcal{A} = \{(p_o, \pi) \in R^L_{++} \times R^J : \exists p_s \in R^L_{++}, s \in S \text{ such that } (p, \pi) \in \mathcal{A}\}$$

  If there is no confusion every time that we write $(p, \pi) \in \mathcal{N} \times R^L_{++} \cdots \times R^L_{++}$ it will be understood that (p_o, π) will belong to $\mathcal{N}$ and the prices p_s will belong to R^L_{++}. The prices p_s can be assumed to belong to $\triangle^L_{++}$ without loss of generality since the second-period budget constraint is unchangeable when p_s is replaced by αp_s for all $\alpha > 0$;
- It is easy to check that for any $p \in R^{SL}_{++}$ the set $\mathcal{N}$ is a convex cone. Thus, so is the set

  $$\mathcal{N} \times (\triangle^L_{++} \times \ldots \triangle^L_{++}) \subset R^{(S+1)L}_{++} \times R^J_+$$

  To abbreviate the notation we will write $\triangle_{++}$ instead of $\triangle^L_{++} \times \ldots \triangle^L_{++} \subset R^{SL}_{++}$;
- Finally, since utility functions are assumed to be strictly increasing, commodity prices equal to zero will be ruled out by assumption. This and the fact that $p_s \in \triangle^L_{++}$ imply that D^j_s is strictly positive and far from zero. This together with (3) in turn implies that $\pi_j >> 0$. In fact:

  Define the sets $L_1 = \{l \in L : A^j_{sl} > 0\}, L_2 = \{l \in L : C_{jl} > 0\}$, and $L^l_3 = \{l' \in L : Y^l_{sl'} > 0\}$. These sets are all nonempty by (A3), (A5) and (A4) respectively. Since $p_s \in \triangle^L_{++}$, one easily has that $p_s Y_s C_j \geq \min_{l\in L}\{(\min_{l'\in L_3} Y^l_{sl'})\}(\min_{l\in L_2} C_{jl}) > 0$. Using that previous same fact, one has that $p_s A^j_s \geq \min_{l\in L_1} A^j_{sl} > 0$. Define α_{sj} to be the minimum between $\min_{l\in L}\{(\min_{l'\in L_3} Y^l_{sl'})\}(\min_{l\in L_2} C_{jl})$ and $\min_{l\in L_1} A^j_{sl} > 0$. This parameter, α_{sj}, is strictly positive. From definition D^{nj}_s, it then follows that $0 < \alpha_{sj} \leq D^j_s$.

Lemma 1 (Individual budget set)

- $B^h : R_+^{L(S+1)} \times R_+^J \to R_+^{L(S+1)} \times R_+^{2J}$ *is a closed correspondence;*
- $B^h(p,\pi)$ *is compact set for* $(p,\pi) \in \mathcal{N} \times \triangle_{++}$*;*
- B^h *is lower hemi-continuous at every* $(p,\pi) \in \mathcal{N} \times \triangle_{++}$.

Proof

- It is very easy to check that the graph of budget correspondence B^h is closed. Therefore, B^h is a closed correspondence;
- It follows from Theorem 2 in Orrillo (2005);
- We first prove that the interior of the budget correspondence B^h denoted by $B^{\circ h}$ is lower hemi-continuous. For every $(p,\pi) \in \mathcal{N} \times \triangle_{++}$ the correspondence $B^{\circ h}$ consists of all (x,θ,φ) satisfying (1) and (2) with strict inequality. In fact, from (2) it follows that $B^{\circ h}$ is nonempty since $(0,0,0) \in B^{\circ h}(p,\pi), \forall (p,\pi) \in \mathcal{N} \times \triangle_{++}$. Let $\lim_{n\to\infty}(p^n,\pi^n) = (p,\pi)$ and $(x,\theta,\varphi) \in B^{\circ h}(p,\pi)$. Then, for every sequence $(x^n,\theta^n,\varphi^n) \to (x,\theta,\varphi)$ and for n large enough, one has

$$p_o^n x_o^n + \pi^n\theta^n + (p_o^n C - \pi^n)\varphi^n < p_o^n \omega_o^h$$

$$p_s^n x_s^n + D_s^n \varphi^n < p_s^n \omega_s^h + D_s^n \theta^n + p_s^n Y_s C \varphi^n + p_s^n Y_s x_o^n, s \in S,$$

  where $D_s^{nj} = \min\{p_s^n A_s^j, p_s^n Y_s C_j\}$. Thus $(x^n,\theta^n,\varphi^n) \in B^{\circ h}(p^n,\pi^n)$ for n large enough, which implies that $B^{\circ h}$ is lower hemi-continuous. Then the result follows from Hildenbrant (1974), page 26, fact 4.

Lemma 2 (Individual demand correspondence)

- Ψ^h *is non-empty-, compact-, convex-valued correspondence;*
- Ψ^h *is upper hemi-continuous at every* $(p,\pi) \in \mathcal{N} \times \triangle_{++}$*;*
- *If the sequence* $\{p^n,\pi^n\}_{n\in N} \subset \mathcal{N} \times \triangle_{++}$ *converges to* (p,π) *such that* p *is not strictly positive or* π *admits arbitrage opportunities, then*

$$\inf\{||(x_o + C\varphi, x_{-o})|| : (x,\theta,\varphi) \in \varphi^h(p^n,\pi^n) \text{ for some } \theta,\varphi\} \to_n +\infty$$

Proof

- First, Ψ^h is nonempty, since U^h is continuous and $B^h(p,\pi)$ is compact $\forall (p,\pi) \in \mathcal{N} \times \triangle_{++}$. Now to show that Ψ^h has compact values, it is sufficient to prove that Ψ^h has closed values, since $B^h(p,\pi)$ is compact for all $(p,\pi) \in \mathcal{N} \times \triangle_{++}$. Closedness of Ψ^h follows from closedness of B^h and continuity of U^h;

- Let $(x^n, \theta^n, \varphi^n) \in B^h(p^n, \pi^n)$ be such that $(x^n, \theta^n, \varphi^n) \to (x, \theta, \varphi)$, and $(p^n, \pi^n) \to (p, \pi)$. We must then prove that $(x, \theta, \varphi) \in \Psi^h(p, \pi)$. Since $B^h(p, \pi)$ is closed, $(x, \theta, \varphi) \in B^h(p, \pi)$. It remains to be proven that (x, θ, φ) is a maximizer. For that, take any (x', θ', φ'). Since the $B^{\circ h}(p, \pi)$ is nonempty for each $(p, \pi) \in \mathcal{N} \times \triangle_{++}$, then for each $\lambda \in]0, 1[$, $\lambda(x', \theta', \varphi') \in B^{\circ h}(p, \pi)$. On the other hand, from $(p^n, \pi^n) \to (p, \pi)$, it follows that there is a natural number n_o such that

$$p_o^n(\lambda x'_o) + \pi^n(\lambda\theta') + (p_o^n C - \pi^n)(\lambda\varphi') < p_o^n \omega_o^h$$

$$p_s^n(\lambda x'_s) + D_s^n(\lambda\varphi') < p_s^n\omega_s^h + D_s^n(\lambda\theta') + p_s^n Y_s C(\lambda\varphi') + p_s^n Y_s(\lambda x'_o), s \in S,$$

  for all $n \geq n_o$, where $D_s^{nj} = \min\{p_s^n A_s^j, p_s^n Y_s C_j\}$. Thus

$$U^h(x_o^n + C\varphi^n, x_o^n) \geq U^h(\lambda x'_o + C(\lambda\varphi'), \lambda x'_{-o}), \forall n \geq n_o.$$

  Now letting λ go to one and using the continuity of U^h the result follows;

- Assuming the contrary, there exists a sequence, say (x^n, φ^n), such that $(x^n, \varphi^n) \to (x, \varphi)$ and $(x^n, \varphi^n, \theta^n) \in \Psi^h(p^n, \pi^n)$ for some θ^n. Since U^h is strictly increasing, then the second-period constraint holds with equality, that is:

$$p_s(x_s^n - \omega_s^h) - (p_s^n Y_s C - D_s^n)\varphi^n - p_s^n Y_s x_o^n = D_s^n \theta^n, \forall s \in S.$$

  Taking limit as n goes to ∞, we have that the sequence $\{D_s^n\theta^n\}$ converges, and therefore it is bounded. So, there exists a strictly positive K such that $0 \leq D_s^{jn}\theta_j^n \leq D_s^n\theta^n \leq K, \forall j \in J$.

  From Remark 3 of Corollary 1 it follows that $0 < \alpha_{sj} \leq D_s^{nj}$, and therefore $\frac{1}{D_s^{nj}} \leq \frac{1}{\alpha_{sj}}$. Since α_{sj} is independent of n, it follows that the sequence $\{\theta^n\}$ converges to some $\theta \in R_+^J$. As Ψ^h is closed at (p, π), then $(x, \theta, \varphi) \in \Psi^h(p, \pi)$. On the other hand, since $p \geq 0$ is not strictly positive or there exists an arbitrage opportunity, and every commodity is desirable, it follows that $\Psi^h(p, \pi) = \emptyset$, which is a contradiction. ■

Let us now define the total excess-demand correspondence Z in three stages: One first defines the correspondence $Z_o : \mathcal{N} \times \triangle_{++} :\to R^L$ as

$$Z_o(p, \pi) := \sum_{h \in H} [\Psi_o^h(p, \pi) + C\Pi_3\Psi^h(p, \pi) - \omega_o^h]$$

next, for each $s \in S$ the correspondence $Z_s : \mathcal{N} \times \triangle_{++} :\to R^L$ as

$$Z_s(p, \pi) := \sum_{h \in H} \{\Psi_s^h(p, \pi) - \omega_s^h - Y_s[\Psi_o^h(p, \pi) + C\Pi_3\Psi^h(p, \pi)]$$

and finally the correspondence $\mathcal{Y} : \mathcal{N} \times \triangle_{++} \to R^J$ to be

$$\mathcal{Y}(p,\pi) := \sum_{h\in H} \Pi_2 T \circ \Psi^h(p,\pi)$$

where $T : R_+^{L(S+1)} \times R_+^J \times R_+^J \to R_+^{L(S+1)} \times R^J$ is the map defined by $T(x, \theta - \varphi)$, and Π_2, Π_3 are the second and third projection maps (see miscellaneous notation). Using these correspondences we can now define the total excess demand correspondence $Z : \mathcal{N} \times \triangle_{++} \to R_+^{L(S+1)} \times R_+^{L(S+1)} \times R^J$ as

$$Z(p,\pi) := (Z_o(p,\pi), (\Psi_s(p,\pi))_{s\in S}, \mathcal{Y}(p,\pi))$$

The main properties of Z follow from properties of individual demand, and are summarized in the following lemma. But to simplify, the following notations will be introduced. If $(z,y) \in Z(p,\pi)$, there then exists $(x^h, \theta^h, \varphi^h) \in \Psi^h$ such that:

$$z_o = \sum_{h\in H} [x_o^h + C\varphi^h - \omega_o^h], z_s = \sum_{h\in H} [x_s - \omega_s^h - Y_s(x_o^h + C\varphi)], \forall s \in S$$

$$y = \sum_{h\in H} (\theta^h - \varphi^h)$$

Lemma 3 (Total excess demand correspondence)

- *Z is non-empty, compact and convex-valued correspondence;*
- *Z is upper hemi-continuous provided $\omega^h >> 0, \forall h \in H$, and;*
- *Z satisfies the following Walras' Law: $\forall (p,\pi) \in \mathcal{N} \times \triangle_{++}$, and $(z,y) \in Z(p,\pi)$ the following hold*

$$p_o z_o + \pi y = 0 \tag{7}$$

$$p_s z_s = D_s y, s \in S \tag{8}$$

where $D_s \in R_+^J$ whose $j-th$ coordinate is D_s^j.

Proof Item 1 follows from the properties of Ψ^h. Item 2 follows from Item 2 of Lemma 3, and finally, since U^h is strictly increasing, Item 3 follows. ■

If $0 \in Z(p,\pi)$ then clearly $(p,\pi) \in \mathcal{N} \times \triangle_{++}$ is an equilibrium price system, for there would be a consumption-portfolio choice $(x^h, \theta^h, \varphi) \in \Psi^h(p,\pi)$ for each $h \in H$, and all the markets would clear.

4. The Existence Theorem

Our main result is the following

Theorem 2 *Under Assumptions 1-5 the economy $\mathcal{E}$ has an exogenous collateral equilibrium.*

Before starting to prove the existence of equilibrium, we first remark the following: Clearly, if (p, π) is an equilibrium price system then $(p, \pi) \in \mathcal{N} \times \triangle_{++}$. Therefore, following Werner (1985), in the sequel only the price system $(p, \pi) \in \mathcal{N} \times \triangle_{++}$ will be considered.

Define the following sets

$$\Sigma := \{(p_o, \pi) \in \overline{\mathcal{N}} : \sum_{l \in L} p_{ol} + \sum_{j \in J} \pi_j = a_o\}$$

where $\overline{\mathcal{N}}$ is the closure of $\mathcal{N}$.

$$\Sigma_s := \{p_s \in R^L_+ : \sum_{s \in S} p_{sl} = a_s\}, \forall s \in S$$

Without loss of generality we take $a_o = 1$, and $a_s = 1$. In the case in which $\Pi_s \mathcal{A} \cap \Sigma_s = \phi$, and $\mathcal{N} \cap \Sigma = \phi$; we adjust a_s so that the previous intersections are nonempty.

Now, for $n \geq L + J$ we define the following sequences of sets:

$$\Sigma^n := \{(p_o, \pi) \in \Sigma : (p_{ol}, \pi_j) \geq (\frac{1}{n}, \frac{1}{n}), \forall (l, j) \in L \times J\}$$

$$\Sigma^n_s := \{p_s \in \Sigma_s : p_{sl} \geq \frac{1}{n}, \forall l \in L\}, \forall s \in S$$

All these sets are clearly compact and convex, and furthermore

$$\Sigma^\circ = \cup_n \Sigma^n \text{ and } \Sigma^\circ_s = \cup_n \Sigma^n_s$$

where "$\circ$" denotes the topological interior.

Lemma 4 *For all n, there exist $(p^n, \pi^n) \in \Sigma^n \times (\Sigma^n_1 \times \ldots \Sigma^n_S)$ and $(z^n, y^n) \in Z(p^n, \pi^n)$ such that, for all $(p, \pi) \in \Sigma^n \times (\Sigma^n_1 \times \ldots \Sigma^n_S)$,*

$$p_o z^n_o + \pi y^n \leq 0, \text{ and } p_s z^n_s \leq 0, \forall s \in S$$

Proof For each n there is a compact convex set B_n which contains $Z(p, \pi)$ for every $(p, \pi) \in \Sigma^n \times (\Sigma^n_1 \times \ldots \Sigma^n_S)$. For any $(z, y) \in B_n$, let $\mu_n(z, y)$ be the set of prices $(p, \pi) \in \Sigma^n \times (\Sigma^n_1 \times \ldots \Sigma^n_S)$ which maximizes the value of both first-period and second-period demand excesses. That is, $p_o z_o + \pi y$ and $p_s z_s, \forall s \in S$.

The correspondence μ_n has the following properties: (i) it is a nonempty, convex correspondence, and (ii) it is an upper hemi-continuous correspondence. (i) is held because the objective functions are linear (thus continuous and concave), and the sets Σ^n and Σ_s^n are compact. (ii) is true since it has the closed graph and its domain is compact.

To each couple (z, y, p, π) of $B_n \times (\Sigma^n \times (\Sigma_1^n \times \ldots \Sigma_S^n)$, associate the set $\mu_n(z, y) \times Z(p, \pi)$. From properties of μ_n and Z it follows that the correspondence

$$\mu_n \times Z : B_n \times (\Sigma^n \times (\Sigma_1^n \times \ldots \Sigma_S^n) \to B_n \times (\Sigma^n \times (\Sigma_1^n \times \ldots \Sigma_S^n)$$

satisfies the conditions of the fixed-point theorem of Kakutani. Therefore, $\mu_n \times Z$ has a fixed point. That is, there exist $(p^n, \pi^n) \in (\Sigma^n \times (\Sigma_1^n \times \ldots \Sigma_S^n)$, and $(z^n, y^n) \in B_n$, such that $(z^n, y^n) \in Z(p^n, \pi^n)$, and $(p^n, \pi^n) \in \mu_n(z^n, y^n)$.

The latter implies that $\forall (p, \pi) \in (\Sigma^n \times (\Sigma_1^n \times \ldots \Sigma_S^n)$,

$$p_o^n z_o^n + \pi^n y^n \geq p_o z_o^n + \pi y^n \tag{9}$$

$$p_s^n z_s^n \geq p_s z_s^n, s \in S \tag{10}$$

In addition, we have:

$$z_o^n \leq 0 \text{ and } y^n \leq 0 \tag{11}$$

otherwise increasing either the price of some commodity or an asset a little, we will raise the value of first-period total excess demand contradicting the fact that $(p^n, \pi^n) \in \mu_n(z^n, y^n)$. On the other hand, $D_s^n y^n \leq 0$, since $y^n \leq 0$ and $D_s^n \in R_{++}^J$.

By Walras' Law, we have

$$p_o^n z_o^n + \pi^n y^n = 0 \tag{12}$$

$$p_s^n z_s^n = D_s^n y^n, s \in S \tag{13}$$

Using the fact that $y^n \leq 0$, it follows that $p_s^n z_s^n \leq 0, s \in S$. Hence $z_s^n \leq 0$ because otherwise, by the same argument above, p_s^n would not maximize the value of second-period total excess demand.

Replacing (12) in (9) and (13) in (10), and taking into account that $D_s^n y^n \leq 0$ we obtain

$$p_o z_o^n + \pi y^n \leq 0, \text{ and } p_s z_s^n \leq 0$$

as desired. ■

4.1 Proof of theorem 2

We consider the sequences (p^n, π^n) and (z^n, y^n) from Lemma 4. Therefore, $\forall (p, \pi) \in \Sigma^n \times (\Sigma_1^n \times \ldots \Sigma_S^n)$,

$$p_o z_o^n + \pi y^n \leq 0, \quad p_s z_s^n \leq D_s^n y^n, \forall s \in S.$$

Since $\Sigma^n \times (\Sigma_1^n \times \ldots \Sigma_S^n) \subset \Sigma \times (\Sigma_1 \times \ldots \Sigma_S)$, and $\Sigma \times (\Sigma_1 \times \ldots \Sigma_S)$ is compact, the sequence (p^n, π^n) has a subsequence converging to some $(\overline{p}, \overline{\pi}) \in \Sigma \times (\Sigma_1 \times \ldots \Sigma_S)$. To keep things simple, let us denote such a subsequence by the same sequence. We divide the remainder of the proof into the following claims.

- **Claim 1:** The sequence z^n is bounded. As $z_o^n = \sum_{h \in H}(x_o^{hn} + C\varphi^{hn} - \omega_o^h)$ and $z_s^n = \sum_{h \in H}(x_s^{nh} - Y_s\omega_o^h - \omega_s^h) - Y_s z_o^n$, we have that

$$-\sum_{h \in H} \omega_o^h \leq z_o^n \text{ and } -\sum_{h \in H}(Y_s\omega_o^h + \omega_s^h) \leq z_s^n$$

  The latter inequality follows from the fact $-Y_s z_o^n \geq 0$, and this in turn, from the second inequality (11), and the fact that $Y_s \in R_+^{L \times L}$ is a positive matrix. By construction, the sequence $\{z^n\}$ is bounded from above by zero, (see the proof of Lemma 4) . Hence the sequence $\{z^n\}$ is bounded. Therefore without loss of generality we may assume that $\{z^n\}$ converges to some $\overline{z}$;

- **Claim 2:** The price system $(\overline{p}, \overline{\pi}) \in \mathcal{N} \times \triangle_{++}$. This follows from Item 3 of Lemma 2, otherwise the sequence $\{z^n\}$ would be unbounded, contradicting Claim 1;

- **Claim 3:** The sequence $\{y^n\}$ is therefore bounded Suppose the contrary, that is $||y^n|| \to \infty$ as n goes to ∞. The sequence $\frac{1}{||y^n||}y^n$ is bounded, since it has norm equal to 1. Hence, passing to a subsequence if necessary, we can assume that it converges to some $y \in -R_+^J$ with $||y|| = 1$. On the other hand, since $(z^n, y^n) \in Z(p^n, \pi^n)$, and $y^n \leq 0$, Walras' Law implies;

$$p_s^n z_s^n = D_s^n y^n \leq 0. \tag{14}$$

  It is known that the sequence $\{p_s^n z_s^n\}$ converges to $\overline{p}_s \overline{z}_s$. Hence it is bounded from below. Dividing both sides of (14) by $||y^n||$, and taking limit as $n \to \infty$, we obtain

$$\overline{D}_s y = 0, \forall s \in S$$

  where $\overline{D}_s^j = \lim_{n \to \infty} D_s^{nj} = \min\{\overline{p}_s A_s^j, \overline{p}_s Y_s C_j\} > 0$ is the $j-$ th coordinate of the vector $\overline{D}_s$. Consider the vector $\overline{\beta} \in R_{++}^S$, of Corollary 1. Therefore, from the first inequality of the two inequalities of Corollary 1 it follows that

$$\sum_{s \in S} \overline{\beta}_s \overline{D}_s y \leq \overline{\pi} y$$

  As $y \in -R_+^J$, it follows that $\overline{\pi} y \leq 0$, and therefore $\overline{\pi} y = 0$. From the last equality it follows that $y = 0$ since $\overline{\pi} >> 0$.[3] Thus we have a contradiction

[3]This follows from Remark 3 of Corollary 1.

because y has been taken into the sphere of ratio 1. Therefore without loss of generality we can suppose that the subsequence $\{y^n\}$ converges to some $\overline{y} \in R^J$.

- **Claim 4:** $(\overline{z}, \overline{y}) = (0, 0)$.

  By closedness of Z we obtain $(\overline{z}, \overline{y}) \in Z(\overline{p}, \overline{\pi})$. By Walras' Law we have

$$\overline{p}_o \overline{z}_o + \overline{\pi y} = 0 \tag{15}$$

$$\overline{p}_s \overline{z}_s = \overline{D}_s \overline{y}, s \in S. \tag{16}$$

  By construction (see proof of Lemma 4) the sequences z^n and y^n are bounded above by zero. Since $-R^J_+$ and $-R^J_+$ are closed, it follows that $\overline{z} \leq 0$ and $\overline{y} \leq 0$.

  Therefore from (15) it follows that $\overline{p}_o \overline{z}_o = 0$ and $\overline{\pi}\,\overline{y} = 0$. Using Claim 3, it follows that $\overline{z}_o =$ and $\overline{y} = 0$. Replacing the latter equality in (16) we obtain $\overline{p}_s \overline{z}_s = 0$ and again from Claim 3 it yields $\overline{z}_s = 0, \forall s \in S$. This ends Claim 4 and therefore the proof of the main theorem as well.

5. Conclusions

Our main result in this paper has been the demonstration of the existence of equilibria for an exogenous collateral economy. The nice feature of this result is the good behavior of the total excess demand on the boundary of its domain – contrary to bad behavior in a setting without default or collateral (see Hart (1975)). Its domain consists of all the commodity – asset price systems that do not admit any arbitrage opportunities. We have not analyzed the efficiency problem, since this was already studied by Araújo et al. (2000) in the case of endogenous collateral, which should dominate our exogenous collateral equilibrium in the sense of Pareto.

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