The Use of Martingale Theory for the Superreplication of Exotic Options in Incomplete Markets*

Christian Johannes Zimmer**

Abstract

In this article we show the importance of modern martingale theory for the pricing and hedging of exotic options, especially in incomplete markets. When emitting an exotic option, the seller firstly has to ask himself whether there exists a hedging strategy for this title or not. Especially, when he wants to use a more realistic model than the simple Black-Scholes framework, the answer is not always obvious. We show in this article how to analyze this problem in the case of an exotic option, the Generalized Bermudian Option, which will turn out to be a generalization of the American option.

Resumo

Mostramos nesse artigo a importância da teoria moderna de martingal para o apreçamento e *hedging* de opções exóticas, especialmente em mercados incompletos. Ao emitir uma opção exótica o vendedor primeiramente se pergunta se há ou não uma estratégia de *hedge* para o título. Especialmente, quando ele deseja utilizar um modelo para o ativo subjacente mais realista ao invés da simples abordagem de Black e Scholes, a solução não é tão óbvia. Analisaremos nesse artigo o caso de uma opção exótica, a Bermudiana generalizada, no qual se tornará uma generalização da opção americana.

Key Words: Martingales, Optional Decomposition, Bermudian Options, Superreplication, Incomplete Markets .

JEL Code: G13 .

---

* This paper is based partially on the *Diplomarbeit* of the author at the faculty of Mathematics at the Technical University of Berlin, Germany. A prior version has been presented at the First Brazilian Congress of Finance in São Paulo, 2001, under the title *Generalized Bermudian options in an incomplete market and their connection to the American option.*

** Universidade de São Paulo, School of Economics, Business Administration and Accountancy (FEA-USP), and Bank Itaú, christian.zimmer@itau.com.br

1. Introduction.

The most famous example for the use of the theory of martingales in finance is the formula of Black-Scholes-Merton for the pricing of options[1]. In this case the price of the underlying asset follows the model of a Wiener process, which, according to Levy's theorem, is a martingale. The advantage of this modeling is that the pricing formula for options depending on this asset can be solved in a quite elegant way via the Feynman-Kac formula instead of analyzing partial differential equations. In addition, the martingale theory offers useful tools especially to attack problems arising in incomplete markets. One of these tools are the decomposition theorems, which we will apply in this paper to determine the price of an exotic option, the Generalized Bermudian Option.

We will analyze under which conditions it will be possible to hedge the payoff of the Bermudian option. When we can show (and we will) that there exists an optional decomposition of the value process of this option, we can deduce the existence of a hedging strategy. The condition for this existence is that the value process of the option is a supermartingale under all (equivalent) measures Q under which the underlying price process is a local martingale.

The specific problems of the option (in this case a put option) under consideration are caused by the following features:

1. The holder of the option can execute his right of selling in several, predefined times, but he does not have to sell.
2. The payoff process is adapted to the possible execution times.
3. The underlying price process $S = (S_t)_{t \geq 0}$ is not only defined at the execution times, but also between them. This implies

[1] See Black & Scholes and recently, Vieira Neto (1973) & Valls Pereira (2002) for a review of some main results of martingale theory for finance, as well as an insight on the wide range of application.

that we can not restrict our analysis to the execution times only. This simplification, many times used in the analysis of (simple) Bermudian options would lead to arbitrage opportunities in our case. What we have to do is to hedge between execution times, too.

We will start our analysis with some further results on martingale theory that have not yet been covered in Vieira Neto & Valls Pereira (2002), thus preparing the tools for our further analysis. Our model is a filtered probability space $\left(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, P\right)$, where the filtration satisfies the *usual conditions*, i. e. it is right-continuous and $\mathcal{F}_0$ contains all P-null sets[2]. In this space lies our (discounted) price process $S = (S_t)_{t \geq 0}$. We assume this price process to be locally bounded and right-continuous with limits on the left ($RCLL$). It won't matter for us whether the process has dimension 1 or d. For clarity, we thus set $S_t : (\Omega, \mathcal{F}_t) \rightarrow \left(\mathbb{R}^+, \mathcal{B}\left(\mathbb{R}^+\right)\right), \forall t \in [0, \infty)$. Moreover it is not an important restriction to assume that $\mathbb{F}$ was created by S. At the starting time $t_0 = 0$ let $\mathcal{F}_{t_0} = \mathcal{F}_0$ be trivial.

Throughout the paper, a process $X = (X_t)_{t \geq 0}$ will be called positive, if $X_t \geq 0$ almost everywhere for all $t \geq 0$. The analog notations will be used for functions, etc.

Definition 1.1 (Local martingale measure) *A probability measure Q will be called a local martingale measure with respect to a process S, if Q is equivalent to P and S is a local martingale with respect to Q.*

Call $\mathbb{M}(S)$ the set of local martingale measures with respect to S. Assume $\mathbb{M}(S) \neq \emptyset$, then the density process of any $Q \in \mathbb{M}(S)$ with respect to a measure $Q^* \in \mathbb{M}(S)$ is given by

[2] For an interpretation, see Vieira Neto & Valls Pereira (2002, p. 358).

$z^{Q^*} = \left(z_t^{Q^*}\right)_{t\geq 0} = \left(\left.\frac{dQ}{dQ^*}\right|_{\mathcal{F}_t}\right)_{t\geq 0}$. The *set of density processes* with respect to Q^* starting at the fixed time $t \geq 0$ will be noted by

$$\mathcal{Z}_t^{Q^*} = \left\{ z \middle| \, z_s = 1, \forall s \leq t; z_s = \left.\frac{dQ}{dQ^*}\right|_{\mathcal{F}_s}, \forall s > t, \text{ for a } Q \in \mathbb{M}(S) \right\}.$$

To emphasize the fact that the process starts at time t the following notation will be used:
$z^{Q^*} =: z^{Q^*,\geq t} \in \mathcal{Z}_t$. If it is important to consider a process from t until the time $u \geq t$ we use the notation (again with respect to Q^*)

$$\begin{aligned} z_u^{Q^*,t} &:= z^{Q^*,\geq t,\leq u} \in \mathcal{Z}_t^{Q^*,u} \\ &= \left\{ z \in \mathcal{Z}_t^{Q^*} \middle| \, z_s = \left.\frac{dQ}{dQ^*}\right|_{\mathcal{F}_s}, \forall s \in (t,u], \text{ for a } Q \in \mathbb{M}(S) \right\}. \end{aligned}$$

Both notations will be used equivalently. If the process is observed at a time $s \in [u,t]$ (e.g. being a random variable), it will be called $\left(z_u^{Q^*,t}\right)_s := \left(z^{Q^*,\geq t,\leq u}\right)_s := z^{Q^*,\geq t,\leq u}\big|_s$, where the notations are once again used equivalently.

From now on, we follow the idea of Kramkov (1996) who assumes that the original measure P is also in $\mathbb{M}(S)$ and that $\mathbb{M}(S) \neq \emptyset$. If it is clear that the reference measure is P, the index of the measure will be dropped $\mathcal{Z}_t^P =: \mathcal{Z}_t$, $z^P =: z$.

A process X, which is a Q-supermartingale (resp. a Q-martingale) for all $Q \in \mathbb{M}(S)$, is called a *universal* or $\mathbb{M}$-supermartingale (resp. universal or $\mathbb{M}$ -martingale). If not specified, every measure dependent characteristic refers to the fixed reference measure $P \in \mathbb{M}(S)$.

With $\mathcal{T}$ we denote the set of stopping times with values in $\mathbb{R}^+$. As in this paper the usual case will be the interval $[0,T]\,, T \in \mathbb{R}^+$ we call $\mathcal{T}$ without further specification the set of all stopping times smaller than or equal to T:

$$\mathcal{T} := \mathcal{T}^{\leq T} :=$$

$$\{\tau : (\Omega, \mathcal{F}) \rightarrow \mathbb{R}^+ | \forall \omega \in \Omega : \tau(\omega) \leq T, \forall t \in \mathbb{R} : \{\omega | \tau(\omega) \leq t\} \in \mathcal{F}_t$$

The set of all stopping times with values after $t \in \mathbb{R}^+$ will be called

$$\mathcal{T}^{\geq t} := \{\tau \in \mathcal{T} | \, \forall \omega \in \Omega : \tau(\omega) \geq t\}\,.$$

The set of times to which the stopping times are reduced in the Bermudian case is

$BZ^{(N}$.

dian times). Further, we call

$$\mathcal{T}_{\text{Ber}} := \Big\{ \tau \in \mathcal{T} | \, \forall \omega \in \Omega : \tau(\omega) \in BZ^{(}$$

the set of admissible stopping times.

Hence, all Bermudian stopping times after $t \in \mathbb{R}^+$ are given by

$$\mathcal{T}_{\text{Ber}}^{\geq t} := \Big\{ \tau \in \mathcal{T}^{\geq t} \big| \, \forall \omega \in \Omega : \tau(\omega) \in BZ^{(N} \quad = \mathcal{T}_{\text{Ber}} \cap \mathcal{T}^{\geq t}.$$

Let $\mathcal{T}_{\text{Ber}}^{\geq t_0} = \mathcal{T}_{\text{Ber}}^{\geq 0} = \mathcal{T}_{\text{Ber}}$.

Wong (1996) generalizes the Bermudian option in the way that the set of possible execution times is given by any measurable stopping region $R \subseteq [0,T]\,, \{T\} \in R$. R will be called *feasible* if for all $(s_n)_{n\in\mathbb{N}} \subseteq R$ with $s_n \downarrow s$ follows $s \in R$. R will be called right-continuous if for all $r \in R \setminus \{T\}$ there exists $(r_n)_{n\in\mathbb{N}} \subset R : r_n$ strictly decreasing and $r_n \to r$.

All R-valued stopping times with reference to $(\mathcal{F}_t, t \in [0,T])$ will be noted with $\mathcal{R}$:

$$\mathcal{R} := \{\tau \in \mathcal{T} | \forall\omega \in \Omega : \tau(\omega) \in R\},$$

respectively for $t \in \mathbb{R}^+$:

$$\mathcal{R}^{\geq t} := \{\tau \in \mathcal{T}^{\geq t} | \forall\omega \in \Omega : \tau(\omega) \in R\}.$$

Analogously, for a $\gamma \in \mathcal{T}$:

$$\mathcal{T}^{\geq\gamma}_{\text{Ber}} \mathrel{\dot{=}} \left\{\tau \in \mathcal{T}^{\geq t} | \forall\omega \in \Omega : \tau(\omega) \in BZ^{(N}\right.$$
$$\mathcal{R}^{\geq\gamma} \mathrel{\dot{=}} \{\tau \in \mathcal{T}^{\geq t} | \forall\omega \in \Omega : \tau(\omega) \in R, \tau(\omega) \geq \gamma(\omega)\}.$$

Hence $BZ^{(N}$
general Bermudian option. With $R = [0,T]$ we have the American option. Our main task will be to generalize the approach of Wong (1996) to the incomplete case, but focusing mainly on the case $R = BZ^{(}$

To put it simply, in order to hedge a payoff, we use a price process and a strategy to create a value which is greater than or equal to the payoff. For this, the stochastic integral is needed. In our analysis, we will restrict ourselves to the case of a stochastic integral defined by simple processes[3]. In what follows a (simple) stochastic process H

[3] Here we followed the formation of concepts of Delbaen (1992). Often, one might find the terms *elementary (predictable) process*or *simple integrand*. For the theory of stochastic integration we refer, for example, to Rogers & Williams (1987).

will be called integrable (with respect to S), if the stochastic integral $H \bullet S = \int HdS$ exists. Let the (discounted) *payoff* be given by an adapted positive process $f = (f_t)_{t \geq 0}$. If R is right-continuous, then f should have right-continuous paths. We assume for any R that

$$\sup_{\tau \in \mathcal{R}} \sup_{Q \in \mathbb{M}(S)} E_Q[f_\tau] < +\infty. \tag{1}$$

Furthermore define a portfolio $\Pi = (v, H, C)$ with the consumption process $C = (C_t)_{0 \leq t \leq T}$, measurable, adapted and increasing (with RCLL-paths, if R is right-continuous), $C_0 = 0$, $C(T) < \infty$, the hedging process $H = (H_t)_{0 \leq t \leq T}$ measurable and adapted with $\int_0^T H_t^2 d[S, S]_t < \infty$ P-almost sure and the initial endowment $v \in \mathbb{R}$. The value process belonging to the portfolio Π will be called $V = (V_t)_{t \geq 0}$. A portfolio will be called hedging portfolio if at any time $t \geq 0$ the payoff f_t may be hedged, i. e.: $V_t \geq f_t$ P-almost sure for all $t \geq 0$. For the minimal hedging portfolio $\Pi^{\min}$ for f with the value process $V^{\min} = \left(V_t^{\min}\right)_{t \geq 0}$ let the following be valid

$$V_t \geq V_t^{\min} \geq f_t, \quad \text{almost sure, } \forall t \geq 0$$

for all hedging portfolios Π with the value process V. If there exists a hedging portfolio, the value process will be represented as $V = v + H \bullet S - C$.

If one defines

$$V_t := \operatorname*{ess\,sup}_{Q \in \mathbb{M}(S), \tau \in \mathcal{R}^{\geq t}} E_Q\left[f_\tau | \mathcal{F}_t\right],$$

then the special case $R = [0, T]$ is treated in Kramkov (1996). In this case it was shown that $(V_t)_{t \geq 0}$ is an $\mathbb{M}$-supermartingale and that it has an optional decomposition. So, for the case $R = [0,T]$ there exists a hedging portfolio. It was also shown that $V = V^{\min}$ holds almost surely.

In this work it will be shown that the minimal hedging portfolio exists for the case $R = BZ^{($
is given by

$$V_t^{\min} = V_0^{\min} + \left(H^{\min} \bullet S\right)_t - C_t^{\min}. \tag{2}$$

To simplify the notations, the following abbreviated forms will be introduced: $\widetilde{S}$, $\widetilde{V}$, $\widetilde{H}$, $\widetilde{C}$, $\widetilde{f}$ and $\widetilde{\mathcal{F}}$ will describe the processes (equiv. the σ-algebra) at the points of execution for the case $R = BZ^{(N}$
$\widetilde{S}_i := S_{t_i}$, $\widetilde{V}_i := V_{t_i}$, $\widetilde{H}_i := H_{t_i}$, $\widetilde{C}_i := C_{t_i}$, $\widetilde{f}_i := f_{t_i}$ and $\widetilde{\mathcal{F}}_i := \mathcal{F}_{t_i}$, $\forall i \in \{0,1,...,N\}$, equivalently $\widetilde{S}_i := S_t$ etc. for all $t \in BZ^{($

The index of an execution time related to any time $t \in \mathbb{R}^+$ will be given by:

$$\begin{aligned} &I : \mathbb{R}^+ \rightarrow \{0,1,...,N\}, \\ &t \mapsto I(t) = \inf \{ i \in \{0,1,...,N-1\} \,|\, t_i \leq t < t_{i+1} \} \wedge N. \end{aligned}$$

---

2. The Value Process of a Bermudian Option.

The value process of the Bermudian option is the critical instrument for our work. The seller of the option has to be able to always pay his liability, i. e. the claim of the option sold. If we succeed in identifying the value process of the Bermudian option as an $\mathbb{M}$-supermartingale we will later have the possibility to decompose it optionally and hence to ensure the existence of a hedging strategy.

However, how to identify the value process is not yet obvious. One might assume that the seller of the option only has to hedge the possible payoff along the execution times and does not need to worry about what happens between these times. But as the price process S is continuous in time that approach is not possible. The seller of the option has to adapt the portfolio even between the execution times in a time-continuous way.

We therefore analyze three different value processes: the value process along the execution times, the value process between two execution times and the global process for all times. Their characteristics will be studied and the connections between the processes will be outlined.

2.1 The Value Process Along the Execution Times.

First, let $R = BZ^{($ $\sim$
$\left(\widetilde{f}_t\right)_{t \in BZ^{(N)}}$ only at the execution times. At the time t_N the payoff $\widetilde{f}_{t_N}$ has to be hedged if the holder has not executed his right yet. In this case we have for the value process $\widetilde{U} = \left(\widetilde{U}_t\right)_{t \in BZ^{(N)}}$, that $\widetilde{U}_{t_N} = \widetilde{f}_{t_N}$ almost surely. At the time t_{N-1} the holder of the option will have to decide whether he wants to execute the option and hence receive the payoff $\widetilde{f}_{t_{N-1}}$ or if he prefers to wait until t_N and then get the value $\widetilde{U}_{t_N}$. So, in t_{N-1} the seller of the option must hold the maximum between the possible payoff $\widetilde{f}_{t_{N-1}}$ and the value of the payoff in t_N at the time t_{N-1}:

$$\widetilde{U}_{t_{N-1}} := \max\left\{\widetilde{f}_{t_{N-1}}, \operatorname*{ess\,sup}_{Q \in \mathbb{M}(S)} E_Q\left[\widetilde{U}_{t_N} \middle| \widetilde{\mathcal{F}}_{t_{N-1}}\right]\right\}.$$

For any Bermudian time we define the process $\widetilde{U}$ recursively as

$$\begin{aligned}\widetilde{U}_{t_i} : &= \max\left\{\widetilde{f}_{t_i}, \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S)} E_Q\left[\widetilde{U}_{t_{i+1}}\middle|\widetilde{\mathcal{F}}_{t_i}\right]\right\}, i\in\{0,...,N-1\}\\ \widetilde{U}_{t_N} : &= \widetilde{f}_{t_N}.\end{aligned}$$

It is obvious that the process $\widetilde{U}$ dominates the payoff process $\widetilde{f}$ for all $Q \in \mathbb{M}(S)$. Now we should find a way to describe the value process in a more suitable way in order to better analyze its characteristics. Defining the process $\widetilde{V} = \left(\widetilde{V}_t\right)_{t\in BZ^{(N)}}$ by

$$\widetilde{V}_t := \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}^{\geq t}_{\mathrm{Ber}}} E_Q\left[\widetilde{f}_\tau\middle|\widetilde{\mathcal{F}}_t\right], t\in BZ^{(N)}.$$

we can show that $\widetilde{U}$ and $\widetilde{V}$ coincide, that is $\widetilde{V}_t = \widetilde{U}_t$ holds almost surely for all $t \in BZ^{(N)}$. This equivalence can be proved by using the ideas of Neveu (1972), Proposition VI-1-2.

It is important to know that the value process $\widetilde{V}$ is an $\mathbb{M}$-supermartingale. We will directly give a formulation for this, generalized on the stopping region R.

Theorem 1(Supermartingale property of the value process) *The process $V = (V_t)_{t\in R}$ is for every $Q \in \mathbb{M}(S)$ a Q-supermartingale with respect to $\mathcal{F} = (\mathcal{F}_t)_{t\in R}$. Especially, $\widetilde{V} = \left(\widetilde{V}_t\right)_{t\in BZ^{(N)}}$ is for every $Q \in \mathbb{M}(S)$ a Q-supermartingale with respect to $\widetilde{\mathcal{F}} = \left(\widetilde{\mathcal{F}}_t\right)_{t\in BZ^{(N)}}$.*

Proof. Choose any t_1 and t_2 from R with $t_1 \leq t_2$. Then it is $\mathcal{Z}_{t_2} \subseteq \mathcal{Z}_{t_1}$ and $\mathcal{R}^{\geq t_2} \subseteq \mathcal{R}^{\geq t_1}$. Because of Proposition 3.1 in Zimmer

(2004)[4] the supermartingale property can be deduced:

$$
\begin{aligned}
E_P\left[V_{t_2}\middle|\mathcal{F}_{t_1}\right] &= \underset{z^{\geq t_2}\in\mathcal{Z}_{t_2},\tau\in\mathcal{R}^{\geq t_2}}{\text{esssup}} E_P\left[f_\tau\left(z^{\geq t_2}\right)_\tau\middle|\mathcal{F}_{t_1}\right] \\
&\leq \underset{z^{\geq t_1}\in\mathcal{Z}_{t_1},\tau\in\mathcal{R}^{\geq t_1}}{\text{ess sup}} E_P\left[f_\tau\left(z^{\geq t_1}\right)_\tau\middle|\mathcal{F}_{t_1}\right] \\
&= V_{t_1}, \qquad t_1 \leq t_2.
\end{aligned}
$$

□

2.2 The Value Process between the Execution Times.

Now, let us study the value process between the execution times. For this process we can use the following result.

Theorem 2 *For any $i \in \{0, ..., N-1\}$ and for all $t \in (t_i, t_{i+1}]$,*

$$V_t^i := \underset{Q\in\mathbb{M}(S)}{\text{ess sup}}\, E_Q\left[\widetilde{V}_{i+1}\middle|\mathcal{F}_t\right]$$

is a supermartingale for every $Q \in \mathbb{M}(S)$ with respect to $\mathcal{F}^i := (\mathcal{F}_t)_{t\in(t_i,t_{i+1}]}$. Furthermore, $V^i = \left(V_t^i\right)_{t\in(t_i,t_{i+1}]}$ has an RCLL-version.

Proof. The lemma can be reduced without any problem to the Proposition 4.2 in Kramkov (1996). One just has to set in this proposition the needed positive random number as $\widetilde{V}_{i+1}$ and reduce the time to the interval $(t_i, t_{i+1}]$ for any $i \in \{0, ..., N-1\}$.

[4] This proposition is replicated in the appendix.

What is left to show is the RCLL-property. In order to show this feature, one can use the approach of El Karoui & Quenez (1995) in analogy as $(\Omega, \mathbb{F}, P)$ was assumed to be complete.

□

We can also represent the value process between the execution times in terms of future execution times.

Proposition 2.1 (Alternative Representation between 2 Execution Times) *For any $i \in \{0, ..., N-1\}$ it holds for every $t \in (t_i, t_{i+1}]$ that:*

$$V_t^i = \operatorname*{ess\,sup}_{Q \in \mathbb{M}(S), \tau \in \mathcal{T}_{\text{Ber}}^{\geq t_{i+1}}} E_Q \left[\widetilde{f}_\tau \middle| \mathcal{F}_t \right] \text{ almost surely.}$$

Proof. Substituting the definition we have

$$\begin{aligned} V_t^i &= \operatorname*{ess\,sup}_{Q \in \mathbb{M}(S)} E_Q \left[\widetilde{V}_{i+1} \middle| \mathcal{F}_t \right] \\ &= \operatorname*{ess\,sup}_{Q \in \mathbb{M}(S)} E_Q \left[\operatorname{ess\,sup}_{Q \in \mathbb{M}(S), \tau \in \mathcal{T}_{\text{Ber}}^{\geq t_{i+1}}} E_Q \left[\widetilde{f}_\tau \middle| \widetilde{\mathcal{F}}_{t_{i+1}} \right] \middle| \mathcal{F}_t \right]. \end{aligned}$$

Once again, as in the proof of Proposition 3.1 (see appendix) we define a sequence of tupels of density processes and stopping times, so that we have an increasing sequence of conditional expectations. Then we use the theorem on monotone convergence and get the result

$$V_t^i = \operatorname*{ess\,sup}_{Q \in \mathbb{M}(S), \tau \in \mathcal{T}_{\text{Ber}}^{\geq t_{i+1}}} E_Q \left[E_Q \left[\widetilde{f}_\tau \middle| \widetilde{\mathcal{F}}_{t_{i+1}} \right] \middle| \mathcal{F}_t \right].$$

This term can be written in another form using the features of the conditional expectation:

$$V_t^i = \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\mathrm{Ber}}^{\geq t_i+1}} E_Q\left[\widetilde{f}_\tau \middle| \mathcal{F}_t\right].$$

□

2.3 Combining the Value Processes with a Global Process.

After having given a representation for the value process between two execution times as well as for the process along the execution times $BZ^{(N)}$ we will now develop a global process for every $t \in [0,T]$.

Therefore we first define, as a natural extension to the Proposition 4.3 of Kramkov (1996, p. 467), the process $\overline{V}^{(1)} = \left(\overline{V}_t^{(1)}\right)_{t\in[0,T]}$ through

$$\overline{V}_t^{(1)} = \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\mathrm{Ber}}^{\geq t}} E_Q\left[f_\tau | \mathcal{F}_t\right], t \in [0,T].$$

At the same time, we combine the already analyzed value processes between two execution times and the value process along the execution times to a new process $\overline{V}^{(2)} = \left(\overline{V}_t^{(2)}\right)_{t\in[0,T]}$ by setting

$$\overline{V}_t^{(2)} = \sum_{i=0}^{N-1}\left(\widetilde{V}_i \cdot \mathbf{1}_{\{t=t_i\}} + V_t^i \cdot \mathbf{1}_{\{t\in(t_i,t_{i+1})\}}\right) + \widetilde{V}_N \cdot \mathbf{1}_{\{t=t_N\}}, t \in [0,T]. \tag{3}$$

The equality of these processes is given by the following

Theorem 3 (Representation of the global Process) *For every $t \in [0,T]$ it holds almost sure that*

$$\overline{V}_t^{(1)} = \overline{V}_t^{(}\quad .$$

Proof. For all $t \in [t$

$$\overline{V}_t^{(2)} = \sum_{i=0}^{N-1} \left\{ \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\text{Ber}}^{\geq t_{i+1}}} E_Q\left[\tilde{f}_\tau \middle| \mathcal{F}_t\right] \cdot \mathbf{1}_{\{t\in(t_i,t_{i+1})\}} \right.$$
$$\left. + \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\text{Ber}}^{\geq t_i}} E_Q\left[\tilde{f}_\tau \middle| \tilde{\mathcal{F}}_{t_i}\right] \cdot \mathbf{1}_{\{t=t_i\}} \right\} + \tilde{V}_N \cdot \mathbf{1}_{\{t=t_N\}}.$$

Here we can apply Lemma 3.3.2 of Zimmer (2000)[5] and get

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{\text{Ber}}^{\geq t_i}} E_Q\left[\tilde{f}_\tau \middle| \tilde{\mathcal{F}}_{t_i}\right] \cdot \mathbf{1}_{\{t=t_i\}} = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{\text{Ber}}^{\geq t_i}} E_Q\left[\tilde{f}_\tau \middle| \mathcal{F}_t\right] \cdot \mathbf{1}_{\{t=t_i\}}. \quad (4)$$

Thus, we can conclude

[5] This lemma was also replicated in the appendix.

$$
\begin{aligned}
\overline{V}_t^{(2)} &= \sum_{i=0}^{N-1} \Bigg\{ \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\text{Ber}}^{\geq t_{i+1}}} E_Q\left[\widetilde{f}_\tau \middle| \mathcal{F}_t\right] \cdot \mathbf{1}_{\{t\in(t_i,t_{i+1})\}} \\
&+ \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\text{Ber}}^{\geq t_i}} E_Q\left[\widetilde{f}_\tau \middle| \mathcal{F}_t\right] \cdot \mathbf{1}_{\{t=t_i\}} \Bigg\} + \widetilde{V}_N \cdot \mathbf{1}_{\{t=t_N\}} \\
&= \sum_{i=0}^{N-1} \Bigg\{ \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\text{Ber}}^{\geq t_{i+1}}} E_Q\left[\widetilde{f}_\tau \middle| \mathcal{F}_t\right] \cdot \mathbf{1}_{\{t\in(t_i,t_{i+1})\}} \\
&+ \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\text{Ber}}^{\geq t_{i+1}}} E_Q\left[\widetilde{f}_\tau \middle| \mathcal{F}_t\right] \cdot \mathbf{1}_{\{t=t_{i+1}\}} \Bigg\} + \widetilde{V}_0 \cdot \mathbf{1}_{\{t=t_0\}}.
\end{aligned}
$$

Attention should be paid to the fact that we changed the indexation in the second equality:

$$
\begin{aligned}
&\sum_{i=0}^{N-1} \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\text{Ber}}^{\geq t_i}} E_Q\left[\widetilde{f}_\tau \middle| \mathcal{F}_t\right] \cdot \mathbf{1}_{\{t=t_i\}} + \widetilde{V}_N \cdot \mathbf{1}_{\{t=t_N\}} \\
&= \sum_{i=0}^{N-1} \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\text{Ber}}^{\geq t_{i+1}}} E_Q\left[\widetilde{f}_\tau \middle| \mathcal{F}_t\right] \cdot \mathbf{1}_{\{t=t_{i+1}\}} + \widetilde{V}_0 \cdot \mathbf{1}_{\{t=t_0\}}.
\end{aligned}
$$

Now, we take together the terms inside the sum and then we have

$$
\begin{aligned}
\overline{V}_t^{(2)} &= \sum_{i=0}^{N-1} \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\text{Ber}}^{\geq t_{i+1}}} E_Q\left[\widetilde{f}_\tau \middle| \mathcal{F}_{t_i}\right] \cdot \mathbf{1}_{\{t\in(t_i,t_{i+1}]\}} \\
&+ \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\text{Ber}}^{\geq t_0}} E_Q\left[\widetilde{f}_\tau \middle| \mathcal{F}_t\right] \cdot \mathbf{1}_{\{t=t_0\}},
\end{aligned}
$$

which we can write, by using Lemma 3.3.1 of Zimmer (2000), see also in the appendix, and by rewriting some terms, as

$$\begin{aligned}
\overline{V}_t^{(2)} &= \operatorname*{esssup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\mathrm{Ber}}^{\geq t}} E_Q\left[\widetilde{f}_\tau \middle| \mathcal{F}_t\right]\cdot \mathbf{1}_{\{t\in(t_0,t_N]\}}+ \\
&+ \operatorname*{esssup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\mathrm{Ber}}^{\geq t}} E_Q\left[\widetilde{f} \middle| \mathcal{F}_t\right]\cdot \mathbf{1}_{\{t=t_0\}} \\
&= \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\mathrm{Ber}}^{\geq t}} E_Q\left[\widetilde{f}_\tau \middle| \mathcal{F}_t\right]\cdot \mathbf{1}_{\{t\in[t_0,t_N]\}} \\
&= \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\mathrm{Ber}}^{\geq t}} E_Q\left[\widetilde{f}_\tau \middle| \mathcal{F}_t\right], t\in[t_0,t_N] \\
&= \overline{V}_t^{(1)}, t\in[0,T].
\end{aligned}$$

□

From now on, the global process will be called $\overline{V} = \left(\overline{V}_t\right)_{t\in[t_0,t_N]}$, $\overline{V} := \overline{V}^{(2)} = \overline{V}^{(2)}$.

And finally, as one would expect, we get that the global value process $\overline{V}$ is an $\mathbf{M}$−supermartingale with reference to $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$.[6]

Proposition 2.2 (Supermartingale property of the global process) *The process $\overline{V}$ is an $\mathbf{M}$−supermartingale with reference to $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$.*

3. The Existence of a Hedging Strategy and its Explicit Form.

We were able to show that the process $\overline{V}$ is an $\mathbb{M}$-supermartingale with respect to $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$. The idea is now

[6] For the proof, see Proposition 3.5 in Zimmer (2000), replicated in the appendix.

to use Theorem 2.1 of Kramkov (1996) on this process and hence get the existence of an optional decomposition of the universal supermartingale.

Theorem 4 (Kramkov - Optional Decomposition of a Supermartingale) *Let* $\overline{V} = \left(\overline{V}_t\right)_{t\in[0,T]}$ *be positive.* $\overline{V}$ *is an* $\mathbb{M}$-*supermartingale with respect to* $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ *if and only if there exists a predictable process* $\overline{H} = \left(\overline{H}_t\right)_{t\geq 0}$, *integrable with respect to* S *and an adapted increasing process* $\overline{C} = \left(\overline{C}_t\right)_{t\geq 0}$ *so that for every* $t \in [0,T]$ *it holds almost sure:*

$$\overline{V}_t = \overline{V}_0 + \left(\overline{H} \bullet S\right)_t - \overline{C}_t. \tag{5}$$

Though this result provides us with the existence of an optional decomposition and a hedging strategy, it does not give satisfactory insight about the explicit form of the processes developed.

Therefore, the aim of this section is to describe the value process $\overline{V}$ at any given time by a hedging process and a consumption process, which involves the jumps at the possible execution times.

In order to do this, we will first show the existence of an optional decomposition of the value process $\widetilde{V}$ along the execution times and the decomposition of the different value processes V^i between the execution times. After this we will analyze how to compound the partial processes to the desired "explicit" representation.

As shown in Zimmer (2000), sections 4.1 and 4.2, there exist optional decompositions for $\widetilde{V}$, the value process along the execution times, and for V^i, the value process between the execution times for $i \in \{0, ..., N-1\}$.[7]

[7] Because of limited space, these results will not be replicated here.

We will now use these two representations to describe the decomposition of $\overline{V}$ in another form: The hedging process will be separated explicitly from the jumps at the execution times.

To do this, we will first take a closer look at representation (3), which we analyzed in section 2.3:

$$\overline{V}_t = \sum_{i=0}^{N-1} \left(\widetilde{V}_i \cdot \mathbf{1}_{\{t=t_i\}} + V_t^i \cdot \mathbf{1}_{\{t \in (t_i, t_{i+1})\}} \right) + \widetilde{V}_N \cdot \mathbf{1}_{\{t=t_N\}}.$$

Into this representation we substitute the optional decompositions for $\widetilde{V}$ and V^i:

$$\begin{aligned} \overline{V}_t = \sum_{i=0}^{N-1} & \left(\widetilde{V}_0 + (H \bullet S)_{t_i} - \widetilde{C}_{t_i} \right) \cdot \mathbf{1}_{\{t=t_i\}} \\ & + \left(V_{t_i}^i + \left(H^i \bullet S \right)_t - C_t^i \right) \cdot \mathbf{1}_{\{t \in (t_i, t_{i+1})\}} \\ & + \left(\widetilde{V}_0 + (H \bullet S)_{t_N} - \widetilde{C}_{t_N} \right) \cdot \mathbf{1}_{\{t=t_N\}}. \end{aligned}$$

Assuming that $t \in [t_i, t_{i+1}]$ for any $i \in \{0, ..., N-1\}$, we would have for $t = t_i$:

$$\overline{V}_{t_i} = \widetilde{V}_{t_i} = \widetilde{V}_0 + (H \bullet S)_{t_i} - \widetilde{C}_{t_i}.$$

In addition, we also have

$$V_{t_i}^i = \operatorname*{ess\,sup}_{Q \in \mathbb{M}(S)} E_Q \left[\widetilde{V}_{i+1} \middle| \mathcal{F}_{t_i} \right],$$

so that we can set

$$V_{t_i}^i = \widetilde{V}_{t_i} - \max\left\{\widetilde{f}_{t_i} - \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S)} E_Q\left[\widetilde{V}_{i+1}\middle|\mathcal{F}_{t_i}\right], 0\right\}$$

The difference between $V_{t_i}^i$ and $\widetilde{V}_{t_i}$ is caused only by the (possible) jump at the execution time

$$\widehat{\delta}_{t_i} := \max\left\{f_{t_i} - \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S)} E_Q\left[\widetilde{V}_{i+1}\middle|\mathcal{F}_{t_i}\right], 0\right\}.$$

This jump might be interpreted as additional consumption for the seller of the option, which only arises if the holder does not execute it optimally. Setting

$$A_{t_i} := \{\text{“the holder of the option does not execute his right in } t_i\text{“}\}, i \in \{0, \ldots, N\},$$

as an event that is independent of S, we have to take a closer look at $\delta_{t_i} = \widehat{\delta}_{t_i} \cdot \mathbf{1}_{A_{t_i}}$.

We want to assure the final state $\widetilde{V}_{t_N} = \widetilde{f}_{t_N}$ so it is reasonable to award the possible consumption δ_{t_i} in t_i to the interval $[t_i, t_{i+1}]$. With $\left(H^i \bullet S\right)_t = 0$ and $C_t^i = 0$ for $t = t_i$, $i \in \{0, ..., N-1\}$, we then have for $t \in [t_i, t_{i+1})$:

$$\begin{aligned}\overline{V}_t &= \widetilde{V}_{t_i} - \delta_{t_i} + \left(H^i \bullet S\right)_t - C_t^i \\ &= \widetilde{V}_0 + (H \bullet S)_{t_i} - \widetilde{C}_{t_i} - \delta_{t_i} + \left(H^i \bullet S\right)_t - C_t^i.\end{aligned} \tag{6}$$

We therefore interpret $\overline{V}_t$ as the value that we hold, if we

1. produce the value $\widetilde{V}_{t_i}$ until t_i,

2. then we realize an additional consumption because of the sub-optimal execution of the holder and then
3. adapt from time t_i until time t the value of our portfolio using $\left(H^i \bullet S\right)_t - C_t^i$.

Our next step in the derivation of an explicit representation is to substitute $\widetilde{V}_{t_i}$. This value may be generated in the following way: Beginning at the previous Bermudian time t_{i-1}, $i \in \{1, ..., N-1\}$, we first realize once again a possible consumption with value $\delta_{t_{i-1}}$, then we adapt the portfolio value until t_i using $\left(H^{i-1} \bullet S\right)$ and C^{i-1}. We start the whole procedure with the value $\widetilde{V}_{t_{i-1}}$:

$$\widetilde{V}_{t_i} = \widetilde{V}_{t_{i-1}} - \delta_{t_{i-1}} + \left(H^{i-1} \bullet S\right)_{t_i} - C_{t_i}^{i-1}.$$

Now we continue this process recursively and get for any $t \in$ [0

$$\begin{aligned} \overline{V}_t &= \widetilde{V}_0 - \delta_0 + \sum_{i=1}^{I(t)} \left\{ \left(H^{i-1} \bullet S\right)_{t_i} - C_{t_i}^{i-1} - \delta_{t_i} \right\} \\ &\quad + \left(H^{I(t)} \bullet S\right)_t - C_t^{I(t)} \\ &= \widetilde{V}_0 - \left(\sum_{i=0}^{I(t)} \delta_{t_i} + \sum_{i=1}^{I(t)} C_{t_i}^{i-1} + C_t^{I(t)} \right) \\ &\quad + \left(\sum_{i=1}^{I(t)} \left(H^{i-1} \bullet S\right)_{t_i} + \left(H^{I(t)} \bullet S\right)_t \right). \end{aligned} \tag{7}$$

With $\overline{V}_0 = \widetilde{V}_0$ and

$$\overline{C}_t \doteq \sum_{i=0}^{I(t)} \delta_{t_i} + \left(\sum_{i=1}^{I(t)} C_{t_i}^{i-1} + C_t^{I(t)} \right),$$

$$\left(\overline{H} \bullet S\right)_t \doteq \sum_{i=1}^{I(t)} \left(H^{i-1} \bullet S\right)_{t_i} + \left(H^{I(t)} \bullet S\right)_t$$

we can deduce the representation of the (global) optional decomposition from (5):

$$\overline{V}_t = \overline{V}_0 + \left(\overline{H} \bullet S\right)_t - \overline{C}_t.$$

4. The Connection between Bermudian, American and European Options.

Not only because of the name, but also because of the configuration of the execution times, the Bermudian option seems to be something between an American and a European option. To finish this paper, we formulate, in a heuristic way, the connections between these options*. The American option is the special case when the execution times are all located on a continuous time line:

$$R = [t_0, t_N] = [0, T] \subset \mathbb{R}^+.$$

Hence, the difference between two execution times tends to zero: $\Delta t_{i+1} = t_{i+1} - t_i \longrightarrow 0$. This leads us to the following assumptions

*The mathematical proofs of the intuitions stated are quite complex and hence not treated in this paper. The interested reader is referred to Zimmer (2000), §5.2 and §5.3.

$$\widetilde{V}_{i+1} - \widetilde{V}_i \rightarrow dV_{t_{i+1}}$$
$$\widetilde{H}_{i+1} \cdot \left(S_{t_{i+1}} - S_{t_i}\right) \rightarrow H_{t_{i+1}} dS_{t_{i+1}}$$
$$\widetilde{C}_{i+1} - \widetilde{C}_i \rightarrow dC_{t_{i+1}}.$$

From now on, we denominate the value processes for the Bermudian, the American and the European option with $V^{\mathrm{Ber}}, V^{\mathrm{Am}}$ and V^{Eu}. In order to emphasize the dependance of the number of execution times, we will use, in this section, the notations $V^{\mathrm{Ber,N}}, N \in \mathbb{N}$ for the value process of the Bermudian option with N execution times. Intuitively, we expect for every $t \in [0, T]$ that:

$$\lim_{N \to \infty} V_t^{\mathrm{Ber,N}} = V_t^{\mathrm{Am}},$$
$$V_t^{\mathrm{Ber,1}} = V_t^{\mathrm{Eu}}.$$

The special feature of the American option compared with the European option is that the holder may execute it before the final time. So, it is reasonable that the prices for these two options differ by an additional premium. We will call this premium *early exercise premium (EEP)*and for every time $t \in [0, T]$ we denote it by EEP_t^{Am}. With this premium we can write the value of an American option at the time $t \in [0, T]$ in the following form[9]:

$$V_t^{\mathrm{Am}} = V_t^{\mathrm{Eu}} + EEP_t^{\mathrm{Am}}.$$

In addition, the Bermudian option also allows an execution before the final date. In the case of the Bermudian option we call the early exercise premium $EEP^{\mathrm{Ber,N}}$, with $N \in \mathbb{N}$. So we can write the value process in the form:

[9] See for instance Carr et al. (1992)

$$V_t^{\text{Ber,N}} = V_t^{\text{Eu}} + EEP_t^{\text{Ber,N}}, t \in [0\ T]$$

We would now expect that for every $t \in [0, T]$ the following holds:

$$\lim_{N\to\infty} EEP_t$$

If we call the optimal stopping time after time t in the Bermudian case

$$b_t := \inf\left\{u \in [t, T) \cap BZ^{(N)} \middle| V_u^{\text{Ber,N}} = f_u\right\} \wedge T$$

and for the American case

$$a_t$$

we can write the value process of the Bermudian option as

$$\begin{aligned} V_t^{\text{Ber,N}} &= \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S)} E_Q\left[f_{b_t} \middle| \mathcal{F}_t\right] \\ &= \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S)} E_Q\left[f_{b_t} + f_{t_N}\right. \\ &= \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S)} E_Q\left[f_{t_N}\right. \qquad {}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\text{Ber}}^{\geq t}} \\ &= : V_t^{\text{Eu}} + EEP_t \end{aligned}$$

If we have only one execution time, call it t_N, then we have

$$EEP_t$$

$$\qquad Q \in \mathbb{M}(S), \tau \in \mathcal{T}_{\text{Ber}}^{\geq t_N}$$

$$= \operatorname*{ess\,sup}_{Q \in \mathbb{M}(S)} E_Q \left[f_{t_N} - f_{t_N} \,|\, \mathcal{F}_t \right.$$

and we have $V_t^{\text{Ber},1} = V_t^{\text{Eu}}$.

Representing the value process of the American option as

$$\begin{aligned} V_t^{\text{Am}} &= \operatorname*{ess\,sup}_{Q \in \mathbb{M}(S)} E_Q \left[f_{a_t} \,|\, \mathcal{F}_t \right] \\ &= V_t^{\text{Eu}} + \operatorname*{esssup}_{Q \in \mathbb{M}(S), \tau \in \mathcal{T}^{\geq t}} E_Q \left[f_\tau - f_{t_N} \,|\, \mathcal{F}_t \right] \\ &=: V_t^{\text{Eu}} + EEP_t \end{aligned}$$

we might expect that for every t $V_t^{\text{Ber,N}} \rightarrow^{N \rightarrow \infty} V_t^{\text{Am}}$ holds if and only if EEP_t
this problem better, one might analyze

1. the convergence of the value of the Bermudian option at any fixed time t to the value of an American option at this time (convergence of a random variable) and
2. the convergence in distribution of the value process of the Bermudian option to the value process of an American option (convergence of a random process).

Unfortunately, it is out of the scope of this paper to answer these questions and we have to leave the reader with the intuition of the connection between the different options.

5. Conclusion.

The aim of this paper was to give an introduction to the (generalized) Bermudian option for the case of an incomplete market. We explained how the value process of the Bermudian option can be separated in a process along the execution times and a process between execution times. An optional decomposition which ensures the existence of a hedging strategy was given. In order to underline the importance of the mathematical approach in the area of advanced finance, exact statements were given as long as the methods applied were not too heavy.

Finally, the connection to the better known American and European option was explained from an intuitive point of view in order to clarify why the (generalized) Bermudian option is a generalization of the American option.

6. Appendix.

Proposition 3.1 from Zimmer (2000) For any $\tau_1, \tau_2 \in \mathcal{R}, \tau_1 \leq \tau_2$ the equation

$$E_P\left[V_{\tau_2} \middle| \mathcal{F}_{\tau_1}\right] = \underset{Q\in\mathbb{M}(S),\tau\in\mathcal{R}^{\geq\tau_2}}{\operatorname{esssup}} E_Q\left[f_\tau \middle| \mathcal{F}_{\tau_1}\right]$$

holds almost surely.

Proof. Let $\tau_1, \tau_2 \in \mathcal{R}, \tau_1 \leq \tau_2$ be almost sure, then the random variable

$$\widehat{V}_{\tau_1} = \underset{Q\in\mathbb{M}(S),\tau\in\mathcal{R}^{\geq\tau_2}}{\operatorname{ess\,sup}} E_Q\left[f_\tau \middle| \mathcal{F}_{\tau_1}\right]$$

can be written with help of the density as

$$\widehat{V}_{\tau_1} = \underset{z^{\geq\tau_2}\in\mathcal{Z}_{\tau_2},\tau\in\mathcal{R}^{\geq\tau_2}}{\operatorname{ess\,sup}} E_P\left[f_\tau\left(z^{\geq\tau_2}\right)_\tau \middle| \mathcal{F}_{\tau_1}\right],$$

so that we have to show:

$$E_P\left[V_{\tau_2} \middle| \mathcal{F}_{\tau_1}\right] = \operatorname*{ess\,sup}_{z^{\geq\tau_2} \in \mathcal{Z}_{\tau_2}, \tau \in \mathcal{R}^{\geq \tau_2}} E_P\left[f_\tau\left(z^{\geq\tau_2}\right)_\tau \middle| \mathcal{F}_{\tau_1}\right], \tag{8}$$

almost sure.

The equality (8) will be shown by proving two inequalities.

1. Step: For every τ_1 and τ_2 from $\mathcal{R}$, $\tau_1 \leq \tau_2$ almost surely, $E_P\left[V_{\tau_2} \middle| \mathcal{F}_{\tau_1}\right] \geq \operatorname{ess\,sup}_{z^{\geq\tau_2} \in \mathcal{Z}_{\tau_2}, \tau \in \mathcal{R}^{\geq \tau_2}} \mathrm{E_P}\left[\mathrm{f}_\tau\left(\mathrm{z}^{\geq\tau_2}\right)_\tau \middle| \mathcal{F}_{\tau_1}\right]$ holds almost surely.

Proof (1. Step). The inequality follows easily from the inequality of Jensen and by using the features of the conditional expectation:

$$\begin{aligned} E_P\left[V_{\tau_2} \middle| \mathcal{F}_{\tau_1}\right] &= E_P\left[\operatorname*{ess\,sup}_{z^{\geq\tau_2} \in \mathcal{Z}_{\tau_2}, \tau \in \mathcal{R}^{\geq \tau_2}} E_P\left[f_\tau\left(z^{\geq\tau_2}\right)_\tau \middle| \mathcal{F}_{\tau_2}\right] \middle| \mathcal{F}_{\tau_1}\right] \\ &\geq \operatorname*{ess\,sup}_{z^{\geq\tau_2} \in \mathcal{Z}_{\tau_2}, \tau \in \mathcal{R}^{\geq \tau_2}} E_P\left[E_P\left[f_\tau\left(z^{\geq\tau_2}\right).\right.\right. \\ &= \operatorname*{ess\,sup}_{z^{\geq\tau_2} \in \mathcal{Z}_{\tau_2}, \tau \in \mathcal{R}^{\geq \tau_2}} E_P\left[f_\tau\left(z^{\geq\tau_2}\right)_\tau \middle| \mathcal{F}_{\tau_1}\right]. \end{aligned}$$

2. Step: For every τ_1 and τ_2 from $\mathcal{R}$, $\tau_1 \leq \tau_2$ almost sure, : $E_P\left[V_{\tau_2} \middle| \mathcal{F}_{\tau_1}\right] \leq \operatorname{ess\,sup}_{z^{\geq\tau_2} \in \mathcal{Z}_{\tau_2}, \tau \in \mathcal{R}^{\geq \tau_2}} \mathrm{E_P}\left[\mathrm{f}_\tau\left(\mathrm{z}^{\geq\tau_2}\right)_\tau \middle| \mathcal{F}_{\tau_1}\right]$ holds almost sure.

Proof (2. Step) Let now $(y_m, \sigma_m)_{1 \leq m \leq N}$ be a sequence in $\left(\mathcal{Z}_{\tau_2}, \mathcal{R}^{\geq\tau_2}\right)$, with:

$$V_{\tau_2} = \sup_{m \geq 1} E_P\left[f_{\sigma_m} \cdot (y_m)_{\sigma_m} \middle| \mathcal{F}_{\tau_2}\right].$$

From this sequence we construct another sequence $(z_m, \tau_m)_{1 \leq m \leq N}$ in the following way:

$$(z_1, \tau_1) = (y_1, \sigma_1)$$

and for $1 < m \leq N$:

$$(z_m, \tau_m) = \begin{cases} (z_{m-1}, \tau_{m-1}), & \text{if } E_P\left[f_{\tau_{m-1}} (z_{m-1})_{\tau_{m-1}} \middle| \mathcal{F}_{\tau_2} \right] \\ & \quad \geq E_P\left[\left[f_{\sigma_m} (y_m)_{\sigma_m} \middle| \mathcal{F}_{\tau_2} \right] \right], \\ (y_m, \sigma_m), & \text{if } E_P\left[f_{\tau_{m-1}} (z_{m-1})_{\tau_{m-1}} \middle| \mathcal{F}_{\tau_2} \right] \\ & \quad < E_P\left[f_{\sigma_m} (y_m)_{\sigma_m} \middle| \mathcal{F}_{\tau_2} \right]. \end{cases}$$

This sequence admits $(z_m, \tau_m)_{m \geq 1} \subseteq \left(\mathcal{Z}_{\tau_2}, \mathcal{R}^{\geq \tau_2} \right)$ and the sequence $\left(E_P\left[f_{\tau_m} (z_m)_{\tau_m} \middle| \mathcal{F}_{\tau_2} \right] \right)_{1 \leq m \leq N}$ is increasing. Then we have

$$E_P\left[f_{\tau_m} (z_m)_{\tau_m} \middle| \mathcal{F}_{\tau_2} \right] = \max_{k \leq m} E_P\left[f_{\sigma_k} (y_k)_{\sigma_k} \middle| \mathcal{F}_{\tau_2} \right]$$

and

$$\begin{aligned} \lim_{m \to \infty} \max_{k \leq m} E_P\left[f_{\sigma_k} (y_k)_{\sigma_k} \middle| \mathcal{F}_{\tau_2} \right] &= \sup_{m \geq 1} E_P\left[f_{\sigma_m} \cdot (y_m)_{\sigma_m} \middle| \mathcal{F}_{\tau_2} \right] \\ &= V_{\tau_2}. \end{aligned}$$

By applying the theorem on monotone convergence and again the features of the conditional expectation we arrive at:

$$\begin{aligned}
E_P\left[V_{\tau_2}\middle|\mathcal{F}_{\tau_1}\right] &= E_P\left[\lim_{m\longrightarrow\infty} E_P\left[f_{\tau_m}\left(z_m\right)_{\tau_m}\middle|\mathcal{F}_{\tau_2}\right]\middle|\mathcal{F}_{\tau_1}\right] \\
&= \lim_{m\longrightarrow\infty} E_P\left[E_P\left[f_{\tau_m}\left(z_m\right)_{\tau_m}\middle|\mathcal{F}_{\tau_2}\right]\middle|\mathcal{F}_{\tau_1}\right] \\
&= \lim_{m\longrightarrow\infty} E_P\left[f_{\tau_m}\left(z_m\right)_{\tau_m}\middle|\mathcal{F}_{\tau_1}\right] \\
&\leq \operatorname*{ess\,sup}_{z^{\geq\tau_2}\in\mathcal{Z}_{\tau_2},\tau\in\mathcal{R}^{\geq\tau_2}} E_P\left[f_\tau\left(z^{\geq\tau_2}\right)_\tau\middle|\mathcal{F}_{\tau_1}\right].
\end{aligned}$$

So, the desired equality (8) is proved. □

Lemma 3.3.1 of Zimmer (2000) It is almost surely true that for every $t \in (t_0, t_N]$

$$\sum_{i=0}^{N-1} \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\text{Ber}}^{\geq t_{i+1}}} E_Q\left[f_\tau\middle|\mathcal{F}_t\right]\cdot\mathbf{1}_{\{t\in(t_i,t_{i+1}]\}} = \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\text{Ber}}^{\geq t}} E_Q\left[f_\tau\middle|\mathcal{F}_t\right].$$

Proof. For any $t \in (t_0, t_N]$ we have

$$\begin{aligned}
&\sum_{i=0}^{N-1} \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\text{Ber}}^{\geq t_{i+1}}} E_Q\left[f_\tau\middle|\mathcal{F}_t\right]\cdot\mathbf{1}_{\{t\in(t_i,t_{i+1}]\}} \\
&= \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\text{Ber}}^{\geq t_1}} E_Q\left[f_\tau\middle|\mathcal{F}_t\right]\cdot\mathbf{1}_{\{t\in(t_0,t_1]\}} + \ldots \\
&+ \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\text{Ber}}^{\geq t_N}} E_Q\left[f_\tau\middle|\mathcal{F}_t\right]\cdot\mathbf{1}_{\{t\in(t_{N-1},t_N]\}}.
\end{aligned}$$

Further, we have for any $i \in \{0, ..., N-1\}, t \in (t_i, t_{i+1}]$, that

$$
\begin{aligned}
\mathcal{T}_{\text{Ber}}^{\geq t} &= \left\{ \tau \in \mathcal{T} \,\middle|\, \forall \omega \in \Omega : \tau(\omega) \in \mathcal{T}^{\geq t} \cap BZ^{(N)} \right\} \\
&= \left\{ \tau \in \mathcal{T} \,\middle|\, \forall \omega \in \Omega : \tau(\omega) \in \mathcal{T}^{\geq t} \cap \{t_{i+1}, ..., t_N\} \right\} \\
&\quad \cup \left\{ \tau \in \mathcal{T} \,\middle|\, \forall \omega \in \Omega : \tau(\omega) \in \mathcal{T}^{\geq t} \cap \{t_0, ..., t_i\} \right\}.
\end{aligned}
$$

But, at the same time

$$
\left\{ \tau \in \mathcal{T} \,\middle|\, \forall \omega \in \Omega : \tau(\omega) \in \mathcal{T}^{\geq t} \cap \{t_0, ..., t_i\} \right\} = \emptyset, \text{ für } t \in (t_i, t_{i+1}].
$$

This results in $\mathcal{T}_{\text{Ber}}^{\geq t} = \mathcal{T}_{\text{Ber}}^{\geq t_{i+1}}$.

Furthermore it holds for every $i \in \{0, ..., N-1\}$ by applying Proposition 2.1 that

$$
\operatorname*{ess\,sup}_{Q \in \mathbb{M}(S), \tau \in \mathcal{T}_{\text{Ber}}^{\geq t}} E_Q \left[f_\tau \middle| \mathcal{F}_t \right] \cdot \mathbf{1}_{\{t \in (t_i, t_{i+1}]\}}
$$

$$
= \begin{cases} \operatorname*{ess\,sup}_{Q \in \mathbb{M}(S), \tau \in \mathcal{T}_{\text{Ber}}^{\geq t_{i+1}}} E_Q \left[f_\tau \middle| \mathcal{F}_t \right] & , t \in (t_i, t_{i+1}] \\ 0 & , \text{ else} \end{cases}
$$

$$
= \operatorname*{ess\,sup}_{Q \in \mathbb{M}(S), \tau \in \mathcal{T}_{\text{Ber}}^{\geq t_{i+1}}} E_Q \left[f_\tau \middle| \mathcal{F}_t \right] \cdot \mathbf{1}_{\{t \in (t_i, t_{i+1}]\}} = V_t^i.
$$

This leads us for $t \in (t_0, t_N]$ to

$$
\begin{aligned}
&\sum_{i=0}^{N-1} \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\mathrm{Ber}}^{\geq t_{i+1}}} E_Q\left[f_\tau \middle| \mathcal{F}_t\right]\cdot \mathbf{1}_{\{t\in(t_i,t_{i+1}]\}} \\
&= \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\mathrm{Ber}}^{\geq t}} E_Q\left[f_\tau \middle| \mathcal{F}_t\right]\cdot \mathbf{1}_{\{t\in(t_0,t_1]\}} + \cdots \\
&\quad + \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\mathrm{Ber}}^{\geq t}} E_Q\left[f_\tau \middle| \mathcal{F}_t\right]\cdot \mathbf{1}_{\{t\in(t_{N-1},t_N]\}} \\
&= \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\mathrm{Ber}}^{\geq t}} E_Q\left[f_\tau \middle| \mathcal{F}_t\right]\cdot \left(\mathbf{1}_{\{t\in(t_0,t_1]\}} + \ldots + \mathbf{1}_{\{t\in(t_{N-1},t_N]\}}\right) \\
&= \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\mathrm{Ber}}^{\geq t}} E_Q\left[f_\tau \middle| \mathcal{F}_t\right]\cdot \mathbf{1}_{\{t\in(t_0,t_N]\}} \\
&= \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\mathrm{Ber}}^{\geq t}} E_Q\left[f_\tau \middle| \mathcal{F}_t\right].
\end{aligned}
$$

□

Lemma 3.3.2 of Zimmer (2000) For all $\tau, \sigma \in \mathcal{R}$ it holds almost sure:

$$
\operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),k\in\mathcal{R}^{\geq\tau}} E_Q[f_k|\mathcal{F}_\tau]\cdot\mathbf{1}_{\{\tau=\sigma\}} = \operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),k\in\mathcal{R}^{\geq\sigma}} E_Q[f_k|\mathcal{F}_\sigma]\cdot\mathbf{1}_{\{\tau=\sigma\}}.
$$

Proof. For every $A \in \mathcal{F}_\sigma$ it is true that

$$
\begin{aligned}
&E_P\left[\operatorname*{ess\,sup}_{Q\in\mathbb{M}(S),k\in\mathcal{R}^{\geq\sigma}} E_Q\left[f_k\middle|\mathcal{F}_\sigma\right]\cdot\mathbf{1}_{\{\tau=\sigma\}}\cdot\mathbf{1}_A\right]\\
&=E_P\left[\operatorname*{ess\,sup}_{z\in Z_\sigma,k\in\mathcal{R}^{\geq\sigma}} E_P\left[f_k\cdot\left(z^{\geq\sigma}\right)_k\cdot\mathbf{1}_{\{\tau=\sigma\}}\cdot\mathbf{1}_A\middle|\mathcal{F}_\sigma\right]\right]\\
&=\sup_{z\in Z_\sigma,k\in\mathcal{R}^{\geq\sigma}} E_P\left[f_k\cdot\left(z^{\geq\sigma}\right)_k\cdot\mathbf{1}_{\{\tau=\sigma\}}\cdot\mathbf{1}_A\right],
\end{aligned}
$$

because of Proposition 6 or equation (8) with $\tau_1=\tau_2=0$. Now, we define the density process

$$
z:=\mathbf{1}_{[[0,\sigma]]}+\left.\frac{dQ}{dP}\right|_{\mathbb{F}}\cdot\mathbf{1}_{((\sigma,T]]}
$$

for a $Q\in\mathbb{M}(S)$. We know that $z\in\mathcal{Z}_\sigma$ by definition and hence it holds

$$
\left\{z\cdot\mathbf{1}_{\{\sigma=\tau\}}\middle|\,z\in\mathcal{Z}_\sigma\right\}=\left\{z\cdot\mathbf{1}_{\{\sigma=\tau\}}\middle|\,z\in\mathcal{Z}_\tau\right\} \tag{9}
$$

as well as

$$
\begin{aligned}
&\left\{(z)_k\cdot f_k\cdot\mathbf{1}_{\{\sigma=\tau\}}\middle|\,z\in\mathcal{Z}_\sigma,k\in\mathcal{R}^{\geq\sigma}\right\}\\
&=\left\{(z)_k\cdot f_k\cdot\mathbf{1}_{\{\sigma=\tau\}}\middle|\,z\in\mathcal{Z}_\tau,k\in\mathcal{R}^{\geq\tau}\right\}.
\end{aligned} \tag{10}
$$

With this, we can deduce

$$
\begin{aligned}
&\sup_{z\in Z_\sigma,k\in\mathcal{R}^{\geq\sigma}} E_P\left[f_k\cdot(z)_k\cdot\mathbf{1}_{\{\tau=\sigma\}}\cdot\mathbf{1}_A\right]\\
&=\sup_{z\in Z_\tau,k\in\mathcal{R}^{\geq\tau}} E_P\left[f_k\cdot(z)_k\cdot\mathbf{1}_{\{\tau=\sigma\}}\cdot\mathbf{1}_A\right].
\end{aligned}
$$

As for $A \in \mathcal{F}_\sigma$ holds $A \cap \{\tau = \sigma\} \in \mathcal{F}_\tau$, it follows with the same argumentation as in (9) respectively (10), just the other way round, and by the repeated use of Proposition 6 that

$$\begin{aligned} &\sup_{z \in Z_\tau, k \in \mathcal{R}^{\geq \tau}} E_P \left[f_k \cdot (z)_k \cdot \mathbf{1}_{\{\tau=\sigma\}} \cdot \mathbf{1}_A \right] \\ &= E_P \left[\operatorname*{esssup}_{z \in Z_\tau, k \in \mathcal{R}^{\geq \tau}} E_Q \left[f_k \cdot (z)_k \middle| \mathcal{F}_\tau \right] \cdot \mathbf{1}_{\{\tau=\sigma\}} \cdot \mathbf{1}_A \right]. \end{aligned}$$

Put together, this results in

$$\begin{aligned} &E_P \left[\operatorname*{ess\,sup}_{Q \in \mathbb{M}(S), k \in} E_Q \left[f_k \middle| \mathcal{F}_\sigma \right] \cdot \mathbf{1}_{\{\tau=\sigma\}} \cdot \mathbf{1}_A \right] \\ &= E_P \left[\operatorname*{esssup}_{Q \in \mathbb{M}(S), k \in \mathcal{R}^{\geq \tau}} E_Q \left[f_k \middle| \mathcal{F}_\tau \right] \cdot \mathbf{1}_{\{\tau=\sigma\}} \cdot \mathbf{1}_A \right]. \end{aligned}$$

□

Proposition 3.5 of Zimmer (2000) The process $\overline{V}$ is an $\mathbf{M}-$supermartingale with reference to $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$.

Proof. Let $i \in \{0, ..., N-1\}$ be as you want and $s, t \in [0, T]$ with $s < t$. We will analyze two cases. Either both times are within the same interval or they are within different intervals, which not necessarily have to be adjacent.

1. Case $s, t \in (t_i, t_{i+1}]$

It holds for every $i \in \{0, ..., N-1\}$

$$\begin{aligned}\overline{V}_t &= \underset{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\text{Ber}}^{\geq t}}{\text{esssup}} E_Q\left[f_\tau|\,\mathcal{F}_t\right] \\ &= \underset{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\text{Ber}}^{\geq t_{i+1}}}{\text{ess sup}} E_Q\left[f_\tau|\,\mathcal{F}_t\right] = V_t^i.\end{aligned}$$

Following Theorem 2, V^i is an $\mathbb{M}$–supermartingale with reference to $\mathcal{F}^i = \left(\mathcal{F}_t^i\right)_{t\in(t_i,t_{i+1}]}$ for all i. We then get for any $i \in \{0,...,N-1\}$

$$\begin{aligned}E_P\left[\overline{V}_t\middle|\,\mathcal{F}_s\right] &= E_P\left[V_t^i\middle|\,\mathcal{F}_s\right] \\ &\leq V_s^i = \underset{Q\in\mathbb{M}(S),\tau\in\mathcal{T}_{\text{Ber}}^{\geq s}}{\text{ess sup}} E_Q\left[f_\tau|\,\mathcal{F}_s\right] \\ &= \overline{V}_s.\end{aligned}$$

2. Case: For any $j \in \{0,...,N-1\}$, $i \in \{0,...,N-1\}$, $j > i$ and $s \in [t_i,t_{i+1}]$, $t \in [t_j,t_{j+1}]$ it is $\overline{V}_s = V_s^i$ and $\overline{V}_t = V_t^j$. As an implication of Theorem 2 we have $V_s^i \geq E_P\left[V_{t_{i+1}}^i\middle|\mathcal{F}_s\right]$ almost sure. As $V_{t_{i+1}}^i = \widetilde{V}_{t_{i+1}}$ and as $\widetilde{V}$ is an **M**–supermartingale with reference to $\widetilde{\mathcal{F}} = \left(\widetilde{\mathcal{F}}_t\right)_{t\in BZ^{(N)}}$, following Theorem 1, we can deduce that

$$\begin{aligned}E_P\left[V_{t_{i+1}}^i\middle|\mathcal{F}_s\right] &\geq E_P\left[E_P\left[\widetilde{V}_{t_j}\middle|\widetilde{\mathcal{F}}_{t_{i+1}}\right]\middle|\mathcal{F}_s\right] \\ &= E_P\left[\widetilde{V}_{t_j}\middle|\mathcal{F}_f\right]\end{aligned}$$

using the features of the conditional expectation. Now, we extend $V^j = \left(V_t^j\right)_{t\in[t_j,t_{j+1}]}$ by

$$V_{t_j}^j = \underset{Q\in\mathbb{M}(S)}{\text{ess sup}}\, E_Q\left[\widetilde{V}_{t_{j+1}}\middle|\mathcal{F}_{t_j}\right]$$

$$V_{t_j}^j = \operatorname*{ess\,sup}_{Q \in \mathbb{M}(S)} E_Q \left[\widetilde{V}_{t_{j+1}} \middle| \mathcal{F}_{t_j} \right]$$

in a natural way in the interval $[t_j, t_{j+1}]$. By doing so, the $\mathbb{M}$-supermartingale property with respect to $\mathcal{F}^j := \left(\mathcal{F}_t^j\right)_{t \in [t_j, t_{j+1}]}$ is not affected. Because of $\widetilde{V}_{t_j} \geq V_{t_j}^j$ almost sure, it holds:

$$E_P \left[\widetilde{V}_{t_j} \middle| \mathcal{F}_s \right] \geq E_P \left[V_{t_j}^j \middle| \mathcal{F}_s \right].$$

Finally, we use the $\mathbb{M}$-supermartingale property of V^j with respect to $\mathcal{F}^j$ for any $t \in [t_j, t_{j+1}]$ and so we get

$$\begin{aligned} E_P \left[V_{t_j}^j \middle| \mathcal{F}_s \right] &\geq E_P \left[E_P \left[V_t^j \middle| \mathcal{F}_{t_j}^j \right] \middle| \mathcal{F}_s \right] \\ &= E_P \left[V_t^j \middle| \mathcal{F}_s \right] \\ &= E_P \left[\overline{V}_t \middle| \mathcal{F}_s \right]. \end{aligned}$$

In the second line we used once again the features of the conditional expectation. Altogether, we get the desired result

$$\overline{V}_s \geq E_P \left[\overline{V}_t \middle| \mathcal{F}_s \right].$$

□

Submitted in October 2002. Revised in August 2003.

References

Black, F. & Scholes, M. 1973. "The pricing of options and corporate liabilities. *Journal of Political Economy*, **81**, 637–659.

Carr, P. Jarrow, R. & Myneni, R. 1992. "Alternative characterizations of American put options"". *Mathematical Finance*, **2**(2), 87–106.

Delbaen, F. 1992. "Representing martingale measures when asset prices are continuous and bounded". *Mathematical Finance*, **2**(2), 107–130.

El Karoui, N. & Quenez, M.-C. 1995. "Dynamic programming and pricing of contingent claims in an incomplete market". *SIAM Journal of Control and Computation*, **33**(1), 26–66.

Kramkov, D. O. 1996. "Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets". *Probability Theory and Related Fields*, **105**, 459–479.

Neveu, J. 1972. *Martingales À Temps Discret*. Paris: Masson et Cie.

Rogers, L. C. G. & Williams D. 1987. *Diffusions, Markov Processes, and Martingales*, volume 2: Ito Calculus. John Wiley and Sons, New York.

Vieira Neto, C. A. & Valls Pereira, P. L. 2002. "Review of major results of martingale theory applied to the valuation of contingent claims. *Brazilian Review of Econometrics*, **21**(2), 355–383.

Wong, D. 1996. *Generalized optimal stopping problems and financial markets*. Addison Wesley Longman, Essex.

Zimmer, C. J. 2000. About Bermudian options. Diplomarbeit, Technical University of Berlin, Faculty of Mathematics, Orientator: M. Schweizer.

NOTE

ADRIANO ROMARIZ DUARTE AWARD

In 1994 the Brazilian Econometric Society instituted the Adriano Romariz Duarte Award. Since then, the award has been given every even year to the best article published in the *Brazilian Review of Econometrics*. The next award will be given to the articles published in volumes 21 and 22.

According to the to the rules governing the Award, the Editor will appoint an award commission which will select the winnning article.

A list of previous award winners is:

1994
"Strategic Behavior in Ascending-Price Multiple-Object Auctions", by Flávio Marques Menezes.

1996
"Déficit Público, A Sustentabilidade do Crescimento das Dívidas Interna e Externa, Senhoriagem e Inflação: Uma Análise do Regime Monetário Brasileiro", by Affonso Celso Pastore.
"Mecanismos de Admissão de Candidatos à Instituições. Modelagem e Análise à Luz da Teoria dos Jogos", by Marilda Sotomayor.

1998
"Estimadores Corrigidos para Modelos SUR Não-Lineares", by Gauss M. Cordeiro and Klaus L.P. de Vasconcellos.
"Cournotian Competition under Knigthian Uncertainty", by Sérgio Ribeiro da Costa Werlang and Hugo Pedro Boff.

2000
"The Slippery Slope: Explaining the Increase in Extreme Poverty in Urban Brazil, 1976 – 1996", by Francisco H. G. Ferreira and Ricardo Paes de Barros.

Subscription Form

To become a subscriber to The Brazilian Review of Econometrics you should fill out this form and send it to:

The Brazilian Review of Econometrics –
A/C: Angela Maria Bevilacqua
Praia de Botafogo, 190 – 10º andar – sala 1032
22253-900 Botafogo, Rio de Janeiro, RJ., Brasil
Tel. (5521) 2551-4658
Fax (5521) 2552-4898
sbe@fgv.br – http: //www.sbe.org.br

Subscriptions

US$ 100 / R$ 150,00
Institutions and new subscribers

US$ 50 / R$ 75,00
Renewals for individual subscribers

US$ 25 / R$ 37,50 Students

Name____________________________

Institution____________________________

Address____________________________

City ____________________________

State ____________________________

Postal Code ________________________

Country __________________________

Telephone__________________________

Fax____________________________

E-Mail __________________________

Year of Subscription__________________

Amount in US$ ou R$__________________

Methods of Payment

Checks

Please send a check with the value of your subscription payable to Sociedade Brasileira de Econometria which can be drawn in a Brazilian bank to the SBE office..

After we receive your check we will send you a receipt, as well as the issues of the Review that correspond to your subscription.

Checks and subscription form should be sent to:

SBE The Brazilian Review of Econometrics
A/C Angela Maria Bevilacqua
Praia de Botafogo, 190 – 10° Andar – Sala 1032
22253-900 – Botafogo – Rio de Janeiro, RJ., Brasil
Tel. (5521) 2551-4658 – Fax (5521) 2552-4898
sbe@fgv.br – http: //www.sbe.org.br

Bank Deposit

A bank deposit must be made in the society's account at Banco do Brasil.

Please send us a fax (5521) 2552 4898 with a copy of your deposit slip attached to your subscription form. The fax is your proof of subscription.

Account Information
Banco do Brasil – Rio de Janeiro – RJ Brasil
Agência Botafogo
Número 0287-9
Account 112.842-6

Payment with a Credit Card

Credit card Number_______________________________________
Expiration Date____
Amount____
Name of the Card ____
Signature____
Date ____

We accept Diners, Master Card e Credicard

Send subscription form to

SBE The Brazilian Review of Econometrics – Revista de Econometria
A/C: Angela Maria Bevilacqua
Praia de Botafogo, 190 – 10° andar – sala 1032
22253-900 Botafogo, Rio de Janeiro, RJ., Brasil
Tel. (5521) 2551-4658
Fax (5521) 2552-4898
sbe@fgv.br – http: //www.sbe.org.br

Submission of Manuscripts to the Brazilian Review of Econometrics

1. Four copies of the original manuscript should be sent to:

   Angela Maria Bevilacqua
   Editorial Assistant, Brazilian Review of Econometrics
   EPGE - Fundação Getulio Vargas
   Praia de Botafogo, 190 - 10o. Andar
   Rio de Janeiro, RJ 22250-900 Brazil.

   Two copies of the manuscript should not have the author's name indicated or revealed in any manner. The manuscripts must be accompanied by a submission letter and be written in English or Portuguese. Submission of a paper is held to imply its contents represent original and unpublished work, and that it has not been submitted elsewhere. Manuscripts are always refereed and may be rejected, returned for specified revisions or accepted.

2. Manuscripts should have an opening page containing the title, the author's name(s), address(es), e-mail and affiliation. It should also contain a set of *Key Words and Phrases* , a primary and secondary *Journal of Economic Literature* classification for the manuscript and an *Abstract* in English of at most 150 words.

3. Manuscripts should be typed with double line spacing on standard size paper, preferably 8.5×11 inches, in a font size of 10 or 12 cpi. The typing area of all pages should be no more than 5.5×8.5 inches (14×21.6 cm).

4. Tables should be typed as part of the text, but in such a way as to avoid confusion with text, extra space should be left above and below the table. Tables, including title, legend, label or number, should fit within the standard typing area and should not run over to the next page.

5. Graphs and other numbered figures should be separate from the text, sized to fit within the width and height of the typed pages and be ready for printing.

6. References should be indicated in the text by the author's name and the year of publication in parentheses. All references should then be collected in alphabetical order at the end of the manuscript.

7. Any acknowledgements, including citations to supporting grants and contracts, should be typed as text and placed before the references at the end of the manuscript.

Submissão de Manuscritos para a Brazilian Review of Econometrics

1. Quatro cópias do manuscrito original devem ser enviadas para:

   Angela Maria Bevilacqua
   Editorial Assistant, Brazilian Review of Econometrics
   EPGE - Fundação Getulio Vargas
   Praia de Botafogo, 190 - 10o. Andar
   Rio de Janeiro, RJ 22250-900 Brazil.

   Em duas cópias do manuscrito não deverá constar o nome do autor indicado ou revelado de nenhuma maneira. Os manuscritos deverão ser escritos em inglês ou português e acompanhados por uma carta de submissão. A submissão de um manuscrito implica que seu conteúdo é original, não publicado e que não está submetido para publicação a nenhum outro periódico. Os manuscritos são sempre julgados por pareceristas e podem ser rejeitados, devolvidos para revisões específicas ou aceitos.

2. Os manuscritos devem ter uma página de abertura contendo o título, o(s) nome(s) do(s) autor(es), endereço(s), e-mail e afiliação. Também deverá conter uma série de *Palavras-Chave e Frases* , uma classificacão primária e secundária para o manuscrito de acordo com o sistema do *Journal of Economic Literature* e um *Abstract* em inglês de no máximo 150 palavras.

3. Os manuscritos devem ser digitados com espaço duplo, em papel tamanho padrão, preferencialmente 8.5 × 11 polegadas, com um tamanho de fonte 10 ou 12 cpi. A área total da página a ser digitada não deverá ser maior do que 5.5 x 8.5 polegadas (14 x 21.6 cm).

4. As Tabelas devem ser digitadas como parte do texto, mas de maneira a evitar confusão com o mesmo, espaço extra deve ser deixado acima e abaixo da tabela. Tabelas, incluindo título, legenda, selo ou número, deverão ajustar-se à área de digitação e não deverão passar para a próxima página.

5. Gráficos e outras figuras numeradas devem ser separados do texto, com um tamanho que se ajusta à largura e altura das páginas digitadas para impressão.

6. As referências devem ser indicadas no texto pelo nome do autor e o ano da publicacão entre parênteses. Todas as referências deverão ser grupadas em ordem alfabética no final do manuscrito.

7. Quaisquer agradecimentos, incluindo citações, permissões de apoio e contratos, devem ser digitados no texto e colocados antes das referências no final do manuscrito.