Optimal Consumption and Investment with Hyperbolic Lévy Motion

José Fajardo Barbachan*

Abstract

We solve the intertemporal consumption and investment problem in a continuous time setting assuming that the security prices follow a Hyperbolic Lévy Motion. Using Stochastic Calculus for Lévy processes, we give sufficient conditions for the existence of optimal consumption and investment policies.

Resumo

Resolvemos o problema do consumo e investimento intertemporal num contexto de tempo contínuo, supondo que os preços dos ativos seguem um Processo de Lévy, denominado Movimento de Lévy Hiperbólico. Usando cálculo estocástico para Processos de Lévy, damos condições suficientes para a existência de políticas ótimas de investimento e consumo.

Key Words: Hyperbolic Lévy motion; Incomplete Markets .

JEL Code: G11 .

*Universidade Católica de Brasília, SGAN 916 Mod B. Asa Norte, 70790-160 Brasília, DF BRAZIL

1. Introduction.

The optimal consumption/investment choices of an investor is an important and classical problem in financial economics. In a seminal paper, Merton [1971] solved this problem for a price taking investor in a continuous time setting using stochastic dynamic programming. He provided explicit solutions for an economy with incomplete markets in which security prices follow a geometric Brownian motion, the endowments follow a Poisson process and the investor has a negative exponential utility with an infinite horizon. Svensson and Werner [1993] obtained solutions for the same problem considering that the endowments follow an arithmetic Brownian motion. They used the fact that with this utility the investor's portfolio choices are independent of wealth.

Extensions of this model to incomplete and/or general constrained markets are due to He and Pearson [1991], Karatzas et al. [1991] and Cvitanić and Karatzas [1992]. They solve the problem using Martingale techniques. It is worth noting that under the Geometric Brownian motion assumption for stock prices the characterization of the Equivalent Martingale Measures (EMM) is given by Girsanov's transformation.

Unfortunately, empirical results have shown that this assumption does not hold for the majority of stocks, since they present "fat tails". To circumvent this problem, Fama [1965] and Mandelbrot and Taylor [1967] proposed a *Pareto-stable distribution*, but the Pareto-stable tails seem to be too heavy. Recently, many authors have developed models that try to describe this phenomenum correctly: Eberlein and Keller [1995] and Kulher et al. [1994] suggested Hyperbolic distributions for modeling German stock returns; this distribution seems to fit well the data. Barndorff-Nielsen [1994], using Generalized hyperbolic distributions (GH), showed that this GH fits very well the distribution of (log) returns for Danish stocks. For

an application of this distribution to fit Brazilian data, see Fajardo, Schuschny and Silva [1999].

The goal of this paper is to solve the optimal consumption/investment problem assuming a GH distribution for stock returns called a Hyperbolic distribution. The paper is organized as follows: in Section 2 we introduce the GH distribution and show that there are EMM. In Section 3 we introduce the model. In Section 4 we study the optimization problem and state the main results. In the last sections we give an example and the appendix with some properties of the GH distribution.

2. Generalized Hyperbolic Distributions and Equivalent Martingale Measures.

The GH distribution was introduced by Barndorff-Nielsen [1977] to study the distribution of sand particles. As we mentioned earlier, many authors have used this distribution and its subclasses to model the returns of some stocks. These distributions have interesting properties: they are invariant under margining, conditioning and affine transformations. Many important distributions belong to this class or are the limit of one of its members, for example the Gaussian distribution, Laplace distribution, Student's t-distribution, Gamma distribution, Hyperbolic distribution and the Normal inverse Gaussian distribution (NIG)[1]. The density of the GH distribution is given by

$$gh(x) = a_\lambda(\delta^2 + (x-\mu)^2)^{(\lambda-1/2)/2} K_{\lambda-1/2}\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right) exp(\beta(x-\mu)), \tag{1}$$

[1] See Barndorff-Nielsen and Blæsid [1981].

where

$$a_\lambda = a_\lambda(\alpha,\beta,\delta) = \frac{(\alpha^2-\beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-1/2}\delta^\lambda K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}; x,\mu \in I\!R$$

$$\delta \geq 0, \mid \beta \mid < \alpha \text{ if } \lambda > 0$$
$$\delta > 0, \mid \beta \mid < \alpha \text{ if } \lambda = 0$$
$$\delta > 0, \mid \beta \mid \leq \alpha \text{ if } \lambda < 0$$

and K_λ is a modified Bessel function (see Abramowitz and Stegun [1968]). The parameters μ and δ describe the location and the scale of the distribution. The other parameters do not have a clear meaning, but doing some transformations we can obtain an asymptotic meaning (see the appendix).

On the other hand, this GH distribution has semi-fat tails:

$$gh(x;\lambda,\alpha,\beta,0,1) \sim \mid x \mid^{\lambda-1} e^{-(\pm\alpha-\beta)x} \text{ as } x \to \pm\infty \text{ and } \lambda > 0.$$

Hence we considerably improve the fit of the data using this GH distribution. Now using properties of the Bessel functions, we have that for $\lambda = 1$ we obtain the following distribution, called the Hyperbolic distribution (H):

$$h(x;\alpha,\beta,\delta,\mu) = \frac{\sqrt{\alpha^2-\beta^2}}{2\delta\alpha K_1\left(\delta\sqrt{\alpha^2-\beta^2}\right)} exp\left(-\alpha\sqrt{\delta^2+(x-\mu)^2}+\beta(x-\mu)\right), \quad (2)$$

where $x,\mu \in I\!R$, $0 \leq \delta$ and $\mid \beta \mid < \alpha$. As in Eberlein and Keller [1995], empirical results on stocks returns motivate the choice of a

centered symmetric Hyperbolic distribution,i.e., $\beta = \mu = 0$. In terms of the following parameterization:

$$\zeta = \delta\alpha, \tag{3}$$

we have that (2) is given by:

$$h(x;\zeta,\delta) = \frac{1}{2\delta K_1(\zeta)} exp\left(-\zeta\sqrt{1+(\frac{x}{\delta})^2}\right). \tag{4}$$

We will denote by $(Z_t^{\zeta,\delta})_{t\geq 0}$ the Lévy process generated by the Hyperbolic distribution with density h as in (4), i.e., the process with independent and stationary increments such that $Z_0^{\zeta,\delta} = 0$ and the distribution of $Z_1^{\zeta,\delta}$ has density h. Eberlein and Keller [1995] called $(Z_t^{\zeta,\delta})_{t\geq 0}$ Hyperbolic Lévy motion.

It is easy to see that this process is a Martingale, since it has centered independent increments. And using the stationarity we have:

$$E\left[(Z_t^{\zeta,\delta})^2\right] = tE\left[(Z_1^{\zeta,\delta})^2\right] < \infty$$

since

$$E\left[(Z_1^{\zeta,\delta})^2\right] = \delta^2\frac{K_2(\zeta)}{\zeta K_1(\zeta)}. \tag{5}$$

Moreover, we can verify that $E\left|Z_t^{\zeta,\delta}\right|^p < \infty$, $\forall t \geq 0$ and $(p \geq 1)$.

2.1 Equivalent Martingale Measures.

It is well known[2] that all the infinitely divisible distributions admit the following Lévy-Khintchine representation:

$$\phi(u) = exp\left(i\mu u - \frac{c}{2}u^2 + \int\left(e^{iux} - 1 - iux\mathbf{1}_{[|x|\leq 1]}\right)G(dx)\right), \tag{6}$$

[2]See Jacod and Shirjaev

where ϕ is the characteristic function of the infinitely divisible distribution, μ is the drift, c is the quadratic variation coefficient and G is a positive measure with $\int(x^2 \wedge 1)G(dx) < \infty$. This measure is called the Lévy measure and describes the jumps in the process.

For Hyperbolic Lévy motion, we can compute its characteristic function and obtain

$$\psi(u) = \frac{\alpha}{K_1(\delta\alpha)} \frac{K_1(\delta\sqrt{\alpha^2 + u^2})}{\sqrt{\alpha^2 + u^2}}.$$

Using this expression, Eberlein and Keller [1995] obtained the following Lévy-Khintchine representation:

$$\psi(u) = exp\left(\int \left(e^{iux} - 1 - iux\right) g(x)dx\right), \tag{7}$$

where g is the density of the Lévy measure (see the appendix for more details on the specific form of this density).

We can see from this representation that there is no continuous part ($c = 0$), so this process is purely discontinuous. From Eberlein and Jacod [1997] we have that the set $\mathcal{Q}$ of equivalent Martingale measure (EMM) is nonempty when the interest rate is constant.

2.2 Stock Prices.

Now we can conclude that a good candidate for the stock prices will be a Hyperbolic Lévy motion plus a drift:

$$dP_t = \rho P_{t-} dt + P_{t-} dZ_t^{\zeta,\delta}. \tag{8}$$

We can rewrite ((8)) as $dP_t = P_{t-} dX_t$ with $X_t = \rho t + Z_t^{\zeta,\delta}$. The solution of this equation is given by the Doléans-Dade exponential:

$$P_t = P_0 exp(Z_t^{\zeta,\delta} + \rho t) \prod_{0<s\leq t} (1 + \Delta Z_s^{\zeta,\delta}) e^{-\Delta Z_s^{\zeta,\delta}}. \tag{9}$$

To see the volatility as an explicit parameter, we perform the following change of variable:

$$\delta =: \delta_\zeta = \sqrt{\zeta \frac{K_1(\zeta)}{K_2(\zeta)}}.$$

By (5) we have

$$\sigma^2 = \delta^2 \frac{K_2(\zeta)}{\zeta K_1(\zeta)}.$$

Then we obtain the process $(Z_t^\zeta)_{t\geq 0} := (Z_t^{\zeta,\delta_\zeta})_{t\geq 0} = (Z_t^{\zeta,\delta}/\sigma)_{t\geq 0}$ and $E[(Z_1^\zeta)^2]$

$$dP_t = \rho P_{t-} dt + \sigma P_{t-} dZ_t^\zeta. \tag{10}$$

Here σ is the daily volatility[3]. Now we will solve the optimal consumption/investment problem.

3. Model.

We will consider a financial market M consisting of 2 assets. The first is called *bond* (the riskless asset) and the second is called *stock* (the risky asset). We will denote the bond price and the stock price at each time t by $B(t)$ and $P(t)$, respectively. The evolution of these prices is modeled, respectively, by the following equations:

$$dB(t) = rB(t)dt, \quad B(0) = 1. \tag{11}$$

[3] One problem with this formulation is that the price could be negative. To avoid this problem, we have to impose a restriction on the jumps: $\sigma Z_t^\zeta > -1$ or do the following reformulation:

$$dP_t = P_{t-}\left[\rho dt + \sigma dZ_t^\zeta + \left(e^{\sigma \Delta Z_t^\zeta} - 1 - \sigma \Delta Z_t^\zeta\right)\right].$$

Thus, $P_t = P_0 exp(\rho t + \sigma Z_t^\zeta)$.

$$dP(t) = P(t^-)[\rho dt + \sigma dZ^{\varsigma}(t)], \quad P(0) \in (0, \infty) \tag{12}$$

In this model the sources of risk are modeled by a Hyperbolic Lévy motion $Z^{\varsigma}(t), 0 \leq t \leq T$. The Hyperbolic Lévy motion is defined on a given complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$; we will denote by $\mathbf{F} = \{\mathcal{F}(t), 0 \leq t \leq T\}$ the $\mathbf{P}-$augmentation[4] of the natural filtration generated by Z^{ς}:

$$\mathcal{F}_{Z^{\varsigma}}(t) = \sigma(Z^{\varsigma}(s), 0 \leq s \leq t), \quad 0 \leq t \leq T.$$

The time horizon is considered finite. The interest rate r, the appreciation rate ρ and the volatility σ are assumed to be constants over the whole interval $[0, T]$.

Now we will introduce a small investor (his decisions do not affect the market prices); this economic agent decides at each moment $t \in [0, T]$:

1. how much money $\pi(t)$ he wants to invest in the stock
2. what his cumulative consumption $C(t)$ should be.

Of course these decisions must be made without foreknowledge of future events (non-anticipative). This facts motivates the following definitions, analogous to the ones used for the Brownian motion case (see Karatzas and Shreve [1998]):

[4] The augmented filtration $\mathbf{F}$ is defined by $\mathcal{F}(t) = \sigma(\mathcal{F}_{Z^{\varsigma}}(t) \cup \mathcal{N})$, where $\mathcal{N} = \{E \subset \Omega : \exists G \in \mathcal{F}$ with $E \subseteq G, \mathbf{P}(G) = 0\}$ denotes the set of $\mathbf{P}-$null events.

Definition 1

(i) An $\mathbf{F}-$ *adapted process*[5] $C=\{C(t), 0 \leq t \leq T\}$ *with increasing, right-continuous paths and* $C(0)=0, C(T)<\infty$ *a.s is called a cumulative consumption process.*

(ii) An $\mathbf{F}-$*progressively measurable*[6], $\mathbb{R}-$*valued process* $\pi = \{\pi(t), 0 \leq t \leq T\}$ *with*

$$\int_0^T |\pi(t)\sigma|^2 \, dt + \int_0^T |\pi(t)(\rho - r)| \, dt < \infty, a.s. \tag{13}$$

is called a portfolio process.

If we denote by $X(t)$ the wealth of this agent at time t, then the amount invested in the bond will be $X(t)-\pi(t)$, from this and (11), (12), we obtain the following equation for the wealth:

$$dX(t) = \pi(t)\frac{dP(t)}{P(t^-)} + (X(t)-\pi(t))\frac{dB(t)}{B(t)} - dC(t) \tag{14}$$

$$= \pi(t)\left[\rho dt + \sigma dZ^{\varsigma}(t)\right] + (X(t)-\pi(t))\, r dt - dC(t)$$

$$= rX(t)dt + \pi(t)\left[(\rho - r)dt + \sigma dZ^{\varsigma}(t)\right] - dC(t).$$

consider an initial capital $x \in \mathbb{R}$, i.e., $X(0)=x$. Then the solution

[5] We have said that a process $\{X_t\}$ is adapted with respect to $\mathbf{F}$ if for all $t \in [0,T]$, X_t is an $\mathcal{F}(t)$-measurable random variable.

[6] A $\mathbb{R}^n$ $-$ *valued* process $X=\{X(t): t \in [0,T]\}$ is said to be progressively measurable with respect to the filtration $\mathbf{F}=\{\mathcal{F}(t)\}$ if for every $t \in [0,T]$ the map $(s,w) \to X(s,w)$ from $([0,t]\times\Omega, \mathcal{B}([0,t])\times\mathcal{F}(t))$ into $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ is measurable, where $\mathcal{B}[0,t]\times\mathcal{F}(t)$ denotes the product $\sigma-field$ of the Borel $\sigma-field$ on $[0,T]$ and $\mathcal{F}(t)$.

of this linear stochastic differential equation is:

$$\gamma(t)X(t) = x - \int_0^t \gamma(s)dC(s) + \int_0^t \gamma(s)\pi(s)\left[\sigma dZ^{\zeta}(s) + (\rho - r))ds\right], \quad 0 \leq t \leq T, \tag{15}$$

where

$$\gamma(t) \triangleq e^{-rt}$$

in the discount factor in $\mathcal{M}$.

Definition 2 *For a given $x \in \mathbb{R}$ and (π, C) as above, the process $X(t) = X^{x,\pi,C}(t)$ of (14) or (15) is called the wealth process corresponding to initial capital x, portfolio π and cumulative consumption process C.*

Notice that we are allowing $(X(t) - \pi(t))$ and $\pi(t)$ to take negative values, which means that short-selling of stock and borrowing at interest rate r are permitted. Thus we need to impose some restrictions on the admissible portfolios.

Definition 3 *We say that a given portfolio process $\pi(\cdot)$ is tame, if the associated discounted gain process:*

$$M^{\pi}(t) \triangleq \int_0^t \gamma(s)\pi(s)\left[\sigma dZ^{\zeta}(s) + (\rho - r))ds\right] \tag{16}$$

is a.s. bounded from below by some real constant:

$$\mathbf{P}\left[M^{\pi}(t) \geq q_{\pi}, \forall 0 \leq t \leq T\right] = 1 \text{ for some } q_{\pi} \in \mathbb{R} \tag{17}$$

In the absence of a condition like (17), one investor could construct oubling strategies, i.e., portfolios that attain arbitrarily large values

of wealth with probability one at $t = T$, starting with zero initial capital at $t = 0$ (see Karatzas [1996]).

Definition 4 *A tame portfolio that satisfies:*

$$\mathbf{P}\left[M^{\pi}(T) \geq 0\right] = 1, \quad \mathbf{P}\left[M^{\pi}(T) > 0\right] > 0 \tag{18}$$

is called an "arbitrage opportunity"(or free lunch). We say that a market M *is arbitrage free if no such portfolio exists.*

The free lunch interpretation of (18) is clear: starting with zero initial capital and using the strategy $\pi(\cdot)$, at the end of the period $t = T$, we have $X(T) = X^{0,\pi,0}(T) = B(T)M^{\pi}(T)$, no risk $X(T) \geq 0$ *a.s.* and positive probability of gain $\mathbf{P}\left[X(T) > 0\right] > 0$. So, we need conditions to preclude these arbitrage opportunities. We know that the existence of EMM rules out these opportunities and, as we have seen in the last section, in our model there would be EMM. Therefore, our market is arbitrage-free.

Unfortunately, our model is typically incomplete since our Hyperbolic process is purely discontinuous, so we do not have a unique EMM and the investor will have to choose one of these EMM. In the next section we will show how to choose one of them.

4. Optimization Problem.

In this section we will formalize the individual problem of the investor and give some definitions, and finally we will present the main result.

Definition 5 *A pair* (π, C) *of portfolio consumption processes is called admissible for the initial capital* $x \geq 0$ *when*

$$X(T) = X^{x,\pi,C}(T) \geq 0, \quad a.s. \tag{19}$$

The class of all such pairs will be denoted by $\mathcal{A}(x)$.

Now we will introduce a new probability P^θ:

$$\frac{dP_t^\theta}{dP_t} = \mathcal{Z}^\theta(t) = e^{\{\theta Z_t^\zeta - t log M(\theta)\}}, \tag{20}$$

where $M(\theta) = \int_{-\infty}^{+\infty} e^{\theta x} h(x) dx$. If we choose θ as the solution of the following equation (for a given r)[7]:

$$r = log\left(\frac{M(\theta+1)}{M(\theta)}\right), \tag{21}$$

then we can verify that the process $\hat{P}_t = e^{-rt} P_t$ will be a Martingale under P^θ, i.e., $P^\theta \in \mathcal{Q}$. Moreover, the process will be a Lévy Process under this probability and will be called the Esscher transform of the original process. Now define the following state price density:

$$H^\theta(t) = \gamma(t)\mathcal{Z}^\theta(t), \quad \forall t \in [0,T]. \tag{22}$$

Using (15) and the generalized Itô's lemma for $F(\mathcal{Z}^\theta(t), \gamma(t)X(t)) = H^\theta(t)X(t)$, we obtain:

$$\begin{aligned} d\left(H^\theta(t)X(t)\right) = & -H^\theta(t^-)dC(t) + H^\theta(t^)\pi(t)(\rho - r)dt \\ & + H^\theta(t^-)\pi(t)\sigma dZ^\zeta(t) \\ & + \gamma(t)X(t^-)d\mathcal{Z}^\theta(t) + H^\theta(t)\Delta X(t) \end{aligned} \tag{23}$$

Now applying the genèralized Itô's Lemma to (20), we obtain:

$$d\mathcal{Z}^\theta(t) = \theta\mathcal{Z}^\theta(t^-)dZ^\zeta(t) + \mathcal{Z}^\theta(t^-)\left[e^{\theta\Delta Z^\zeta(t)} - 1 - \theta\Delta Z^\zeta(t)\right] \tag{24}$$

[7] See Gerber and Shiu [1994].

Then replacing (24) in (23), we have

$$d\left(H^{\theta}(t)X(t)\right) = -H^{\theta}(t^{-})dC(t) + H^{\theta}(t^{-})\left[\pi(t)\sigma + \theta X(t^{-})\right]dZ^{\varsigma}(t) + dD(t) \tag{25}$$

where $dD(t)$ is given by

$$dD(t) = H^{\theta}(t^{-})\pi(t)(\rho - r)dt + H^{\theta}(t^{-})X(t^{-})\left[e^{\theta\Delta Z^{\varsigma}(t)} - 1 - \theta\Delta Z^{\varsigma}(t)\right]$$

Now in order to formulate our optimization problems, we need the concept of a utility function.

Definition 6 *We say that a function* $u : (0,\infty) \to \mathbb{R}$ *is a utility function if it is strictly increasing, strictly concave, continuously differentiable and*

$$u'(\infty) \triangleq \lim_{x\to\infty} u'(x) = 0 \quad \textit{and} \quad u'(0_{+}) \triangleq \lim_{x\downarrow 0} u'(x) = \infty \tag{26}$$

.

Remark: Examples of utility functions are $u(x) = \log x$ and $u(x) = \frac{x^{\delta}}{\delta}, \delta \in (-\infty, 1)\backslash\{0\}$.

We will denote by $I(\cdot)$ the inverse of the derivative $u'(\cdot)$; both these functions are continuous, strictly decreasing, and map (0,∞) onto itself with $I(0_{+}) = u'(0_{+}) = \infty, I(\infty) = u'(\infty) = 0$. We shall consider also de convex dual

$$\widehat{u}(y) \triangleq \max_{0<x<\infty}[u(x) - xy] = u(I(y)) - yI(y), 0 < y < \infty. \tag{27}$$

Thus $\hat{u}(\cdot)$ is a convex decreasing function, continuously differentiable on (0,∞) and satisfies

$$\widehat{u}'(y) = -I(y),\ 0 < y < \infty \tag{28}$$

$$u(x) \triangleq \min_{0<y<\infty} [\widehat{u}(y) + xy] = \widehat{u}(u'(x)) + xu'(x) \tag{29}$$

$$\widehat{u}(\infty) = u(0_+),\ \widehat{u}(0_+) = u(\infty) \tag{30}$$

4.1 Optimization without income stream.

Consider a small invest or one who has initial capital $x > 0$ and wants to choose a portfolio $\pi(\cdot)$ and consumption rate process $\{c(t),\ 0 \leq t \leq T\}$, such that $C(t) = \int_0^t c(s)ds$, in order to maximize his expected utility from the terminal wealth $X^{x,\pi,C}(T)$ and from consumption.

Given the utility functions g and $u(t,\cdot)$ as in the above definition with the respective I_g and $I_u(t,\cdot)$, we will define the following classes:

$$\mathcal{A}_g(x) \triangleq \left\{ (\pi, C) \in \mathcal{A}(x) \Big/ Eg^- \left(X^{x,\pi,C}(T)\right) < \infty \right\}$$

$$\mathcal{A}_u(x) \triangleq \left\{ (\pi, C) \in \mathcal{A}(x) \Big/ E \int_0^T u^-(t, c(t))dt < \infty \right\}$$

where $f^-(x) = \max\{-f(x), 0\}$.

Then in order for the investor optimal problem to be well-defined, our small investor will maximize the expected utility from consumption and terminal wealth over the following class:

$$\mathcal{A}_0(x) \triangleq \mathcal{A}_u(x) \cap \mathcal{A}_g(x).$$

The value function of the investor problem will be

$$V_\theta(x) = \sup_{(\pi,C)\in\mathcal{A}_0(x)} E\left[\int_0^T u(t, c(t))dt + g\left(X^{x,\pi,C}(T)\right) \right]. \tag{31}$$

Now to solve this optimization problem, consider in the context of the market model M described above a contingent claim[8] ξ and a consumption process C that satisfy

$$E\left[H^{\theta}(T)\xi + \int_0^T H^{\theta}(t^-)dC(t)\right] = x > 0. \tag{32}$$

If there exists a portfolio process $\pi(\cdot)$ such that $(\pi, C) \in \mathcal{A}_0(x)$ and $X^{x,\pi,C}(T) = \xi$, we can conclude that the optimal problem is in some sense (see Theorem 1 below) equivalent to the problem of maximizing $E\left[\int_0^T u(t, c(t))dt + g(\xi)\right]$ over all pairs (ξ, C) of contingent claims and consumption rate processes that satisfy the constraint (32).

In order to state Theorem 1, we need the following:

Remark

Given $y > 0$ (*Lagrange multiplier*), by (27)

$$E\left[\int_0^T u(t, c(t))dt + g(\xi)\right]$$

$$+y\left[x - E\left[H^{\theta}(T)\xi + \int_0^T H^{\theta}(t^-)c(t)dt\right]\right] =$$

$$= E\left[\int_0^T \left[u(t, c(t))dt - yH^{\theta}(t^-)c(t)\right]dt\right] + E\left[g(\xi) - yH^{\theta}(T)\xi\right] + xy \tag{33}$$

[8] A contingent claim is any random variable $\mathcal{F}_T-$ measurable

$$\leq E\left[\int_0^T \widehat{u}(t, yH^\theta(t^-))dt\right] + E\left[\widehat{g}\left(yH^\theta(T)\right)\right] + xy$$

the equality holds if and only if

$$\xi_\theta = I_g\left(yH^\theta(T)\right) \quad \text{and} \quad c_\theta(t) = I_u\left(t, yH^\theta(t^-)\right). \tag{34}$$

Define

$$\mathcal{X}_\theta(y) \triangleq E\left[H^\theta(T)I_g\left(yH^\theta(T)\right) + \int_0^T H^\theta(t^-)I_u\left(t, yH^\theta(t^-)\right)dt\right]$$

$\mathcal{X}_\theta(\cdot)$ maps (0,∞) onto itself and is continuous, strictly decreasing with

$$\mathcal{X}_\theta(0_+) \triangleq \lim_{y\downarrow 0} \mathcal{X}_\theta(y) = \infty, \mathcal{X}_\theta(\infty) \triangleq \lim_{y\to\infty} \mathcal{X}_\theta(y) = 0.$$

If we denote by $\mathcal{Y}^\theta(\cdot) = \mathcal{X}_\theta^{-1}(\cdot)$, then the Lagrange multiplier $y > 0$ is uniquely determined by

$$y = \mathcal{Y}^\theta(x).$$

Now we can state our main result:

Theorem 1 *Suppose* $x \in (0,\infty)$ *and* $V_\theta(x) < \infty, \forall x \in (0,\infty)$. *For any* $x > 0$, *consider the optimization problem with value function* $V_\theta(x)$ *as in (31) and define* ξ_θ *and* $c_\theta(\cdot)$ *as in (34). Assume that*

(a)there is a portfolio process $\pi_\theta(\cdot)$ *such that* $(\pi_\theta, C_\theta) \in \mathcal{A}(x)$ *and* $X^{x,\pi_\theta,C_\theta}(T) = \xi_\theta$;

(b) the process D_t *is a local Martingale and* $D_0 = 0$.

Then (π_θ, C_θ) *are the solutions of the investor problem and the value function is given by*

$$V_\theta(x) = \mathcal{G}(\mathcal{Y}_\theta(x))$$

where

$$\mathcal{G}(y) \triangleq E\left[\int_0^T u(t, I_u(t, yH^\theta(t^-)))dt + g\left(I_g\left(yH^\theta(T)\right)\right)\right], \quad \forall y \in (0,\infty) \tag{35}$$

Moreover, the convex dual of $V_\theta(\cdot)$ *is*

$$\widehat{V_\theta}(y) = \mathcal{G}(y) - y\mathcal{X}_\theta(y) = E\left[\int_0^T \widehat{u}(t, yH^\theta(t^-))dt\right] + E\left[\widehat{g}\left(yH^\theta(T)\right)\right]$$

Proof

By construction, ξ_θ and c_θ satisfy (32) and using the following inequality

$$f(I_f(y)) \geq f(x) + y\left[I_f(y) - x\right]$$

for every utility function f, we obtain

$$u(t, c_\theta(t)) \geq u(t,1) + \mathcal{Y}_\theta(x)H^\theta(t^-)(c_\theta(t) - 1),\ 0 \leq t \leq T$$

$$g(\xi_\theta) \geq g(1) + \mathcal{Y}_\theta(x)H^\theta(T)(\xi_\theta - 1),\ \ a.s.$$

Therefore,

$$E\left[\int_0^T u^-(t,c_\theta(t))dt + g^-(\xi_\theta)\right] \leq |g(1)| + \int_0^T |u(t,1)|\,dt$$

$$+\mathcal{Y}_\theta(x)\left[H^\theta(T) + \int_0^T H^\theta(t^-)dt\right] < \infty, \tag{36}$$

since $EH^\theta(t) \leq e^{rT}, 0 \leq t \leq T$. And by (a), we have that there exists a portfolio process π_θ with $(\pi_\theta, C_\theta) \in \mathcal{A}(x)$ (also in $\mathcal{A}_0(x)$ thanks to (36)) and $X^{x,\pi_0,C_\theta}(T) = \xi_\theta \ \ a.s.$

Now take an arbitrary $x > 0$, $(\pi, C) \in \mathcal{A}_0(x)$ and $y > 0$. From (b) and (25) we have that the following process is a bounded local martingale:

$$H^\theta(t)X(t) + \int_0^t H^\theta(s^-)dC(s).$$

Thus it is a supermartingale[9] and from (33)

$$E\left[\int_0^T u(t,c(t))dt + g\left(X^{x,\pi,C}(T)\right)\right] \leq$$

$$E\left[\int_0^T u(t,c(t))dt + g\left(X^{x,\pi,C}(T)\right)\right] +$$

[9] A process $\{X_t\}$ is said to be a $\{\mathcal{F}_t\}-$martingale (submartingale, supermartingale) if the following conditions are satisfied:

(i) $E|X_t| < \infty$

(ii) $E[X_t/\mathcal{F}_t] = X_s \ a.s. \ \ \forall s \leq t$ ($\geq$, $\leq$, respectively).

$$+y\left[x - E\left[H^{\theta}(T)X^{x,\pi,C}(T) + \int_0^T H^{\theta}(t^-)dC(t)\right]\right]$$

$$\leq Q(y) + xy \tag{37}$$

where

$$Q(y) \triangleq E\left[\int_0^T \widehat{u}(t, yH^{\theta}(t^-))dt + \widehat{g}\left(yH^{\theta}(T)\right)\right] = \mathcal{G}(y) - y\mathcal{X}_{\theta}(y)$$

In particular, it follows that

$$V_{\theta}(x) \leq Q(y) + xy, \quad \forall x > 0$$

Hence,

$$\widehat{V}_{\theta}(y) \leq Q(y), \quad \forall y > 0. \tag{38}$$

On the other hand, the inequality (37) holds as equality if and only if $y = \mathcal{Y}_{\theta}(x)$ and $(\pi, C) = (\pi_{\theta}, C_{\theta})$ then

$$E\left[\int_0^T u(t, c_{\theta}(t))dt + g\left(X^{x,\pi_0,C_{\theta}}(T)\right)\right] = Q(\mathcal{Y}_{\theta}(x)) + x\mathcal{Y}_{\theta}(x)$$

$$= \mathcal{G}(\mathcal{Y}_{\theta}(x)).$$

Therefore,

$$V_{\theta}(x) = \mathcal{G}(\mathcal{Y}_{\theta}(x))$$

and also

$$Q(y) = V_{\theta}(\mathcal{X}_{\theta}(y)) - y\mathcal{X}_{\theta}(y) \leq \sup_{x>0}\left[V_{\theta}(x) - xy\right]$$

For every $y > 0$, by (38), we obtain $Q(y) = \widehat{V}_\theta(y)$. □

Remarks on the assumptions

• assumption (a) is a necessary and sufficient condition for the optimality of the consumption, since we need to finance it. To look for its existence, we have to solve the following equation:

$$e^{-rT} I_g(\mathcal{Y}^\theta(x) H^\theta(T)) = x - \int_0^T e^{-rs} dC_\theta(s) + \int_0^T e^{-rs} \pi(s) \left[\sigma dZ^\zeta(s) + (\rho - r)) ds\right],$$

but the problem is that we can not assure this solution will be available for trading, since the market is incomplete.

• assumption (b) is a sufficient condition but not necessary. This condition let us to control the jumps of the wealth. In the Brownian motion case this condition is automatically satisfied, as we will see in the example below.

5. Example.

Now take $r = 0$, $u(t,x) = g(x) = logx$, and $\rho = \sigma = 1$. Then $I_u(x) = I_g(x) = \frac{1}{x}$ and

$$\xi_\theta = \frac{1}{yH^\theta(T)} \text{ and } c_\theta(t) = \frac{1}{yH^\theta(t^-)}$$

but

$$x = \mathcal{X}^\theta(y) = E\left[H^{\mathcal{Z}}(T) I_g\left(yH^\theta(T)\right) + \int_0^T H^\theta(t^-) I_u\left(t, yH^\theta(t^-)\right) dt\right]$$

$$= E\left[\frac{1}{y} + \frac{T}{y}\right].$$

Then $y = \frac{T+1}{x}$,

$$\xi_\theta = \frac{x}{(T+1)\mathcal{Z}^\theta(T)} \text{ and } c_\theta(t) = \frac{x}{(T+1)\mathcal{Z}^\theta(t^-)},$$

which implies

$$c_\theta(t) = \frac{x}{(T+1)e^{\{\theta Z^\zeta_{t-} - t log M(\theta)\}}},$$

where θ is the solution of:

$$0 = log\frac{M(\theta+1)}{M(\theta)}$$

Thus $\theta = \alpha - 1$, and

$$M(\theta) = \sqrt{\frac{\alpha}{2\alpha-1}}\frac{K_1(\delta\sqrt{2\alpha-1})}{K_1(\delta\alpha)}$$

In the case of Brownian motion, we obtain

$$c_0(t) = \frac{x}{(T+1)e^{\{-W_t - \frac{t}{2}\}}}.$$

We can observe that jumps in price are implemented in the optimal consumption. Thus positive jumps will produce a diminution on consumption, it is not observed in the Brownian case.

For the optimal investment we know that in the Brownian case we have:

$$\pi_0(t) = \frac{x}{T+1} \frac{1+T-t}{e^{\{-W_t - \frac{1}{2}\}}}$$

and in our case we have to obtain the solution of the following equation:

$$\frac{x}{(T+1)H^{\theta}(T)} = x - C_{\theta} + \int_0^T \pi_{\theta}(s)[dZ_s^{\zeta} + ds]$$

6. Conclusions.

In this paper we have obtained sufficient conditions for the existence of optimal consumption and investment policies assuming that asset price follows a Hyperbolic Lévy motion. As we mentioned earlier, this distribution provides a better fit of the data than the geometric Brownian motion. Many distributions can be used to improve the data fit. In this sense, it would be important to study processes without independent increments, because these processes can be used to model other stylized facts of financial data, such as the observed persistence in the correlations of the absolute and squared log returns, something that can not be performed with Lévy processes (see for instance Rydberg, T. [1999]
considered is the use of a specific EMM, given by the Gerber and Shiu [1994]
cies for whatever EMM and in this way we allow the investor to have a diversified perception of the behavior of the market.

In closing, it would be important to address the existence of equilibrium, since as we know that many recent empirical results have supported the use of alternative models for stock prices, such as the generalized hyperbolic distributions, but without any equilibrium analysis. So we could use these optimality conditions to study

the existence of equilibrium and give theoretical support to these alternative models.

7. Appendix.

7.1 Infinite Divisibility.

A random variable $X \in \mathbb{R}^n$ is said to be distributed with a normal variance-mean mixture with location μ, drift β, structure matrix Δ and mixing distribution F, if there is a random variable z with distribution F on $[0, \infty)$ and the conditional distribution of X under z is normal:

$$P^{X|z} = N_n(\mu + z\beta, z\Delta)$$

$\mu, \beta \in \mathbb{R}^n$, Δ is symmetric, positive definite and det(Δ)=1. This last condition excludes the case $\Delta = 0$ and makes the choice of parameters unique. This distribution will be denoted by $NVMM(\mu, \beta, \Delta, F)$. Notice that any distribution F on $[0, \infty)$ could be written as $NVMM(0, 1, 0, F)$.

The GH distribution can be represented as a Normal Variance-Mean Mixture (NVMM) distribution. To this end, we will use as a mixing distribution a generalized inverse Gaussian distribution $N^-(\lambda, \nu, \psi)$ which has the following density:

$$f(x) = \frac{(\psi/\nu)^{\lambda/2}}{2K_\lambda(\sqrt{\psi\nu})} x^{\lambda-1} exp\{\frac{1}{2}(\nu x^{-1} + \psi x)\}, \quad x > 0.$$

The parameter domain is $\lambda \in \mathbb{R}$, $\nu > 0$, $\psi > 0$. And $\nu = 0$ is allowed for $\lambda > 0$, $\psi = 0$ is allowed for $\lambda < 0$. Then,

$$GH(\lambda, \alpha, \beta, \delta, \mu, \Delta) = NVMM(\mu, \beta\Delta, \Delta, N^-(\lambda, \delta^2, \epsilon^2)) \qquad (39)$$

$$\epsilon^2 = \alpha^2 - < \beta, \Delta\beta >$$

Remember that in (1) if $n = 1$, then $\Delta = 1$. Also Barndorff-Nielsen [1978] showed that the Normal distribution is obtained as a limiting case for $\delta \to \infty$ and $\delta/\alpha \to \sigma^2$.

Now, as Barndorff-Nielsen and Halgreen [1977] showed, N^- distributions are infinitely divisible. Then they used (39) to prove that GH distributions are infinitely divisible[10]. Then they generate a Lévy processes.

7.2 Meaning of Parameters.

One argument against the use of a GH distribution is that its parameters do not have a clear meaning. To circumvent this problem, many parameterizations have been suggested, following Barndorff-Nielsen et al.[1985]:

$$\xi = (1 + \delta\sqrt{\alpha^2 - \beta^2})^{-\frac{1}{2}}, \quad \chi = \frac{\xi\beta}{\alpha} \tag{40}$$

$$\zeta = \delta\sqrt{\alpha^2 - \beta^2}, \qquad \varrho = \frac{\beta}{\alpha}$$

They showed that in the case of a hyperbolic distribution (χ, ξ) may be plotted in a triangle shape, which reflects asymptotically the shape, i.e., skewness and kurtosis of the distribution.

[10] If ψ is the characteristic function of F, then the characteristic function ι of NVMM(μ,β,Δ,F) is

$$\iota(t) = e^{<t,\mu>}\psi\left(<\beta,t> + \tfrac{i}{2}<t,\Delta t>\right).$$

With this equation, we can verify that the sum of k i.i.d NVMM variables with the same β and Δ is a NVMM :

$$NVMM(\mu,\beta,\Delta,F)^{*k} = NVMM(k\mu,\beta,\Delta,F^{*k}).$$

7.3 Existence of Equivalent Martingale Measures.

Eberlein and Keller [1995] obtained the following expression for the density g of the Lévy measure G of the Hyperbolic Lévy motion

$$g(x) = \frac{1}{|x|}\left(\int_0^\infty \frac{e^{-\sqrt{2y+(\zeta/\delta)^2}|x|}}{\pi^2 y\left(J_1^2(\delta\sqrt{2y}) + Y_1^2(\delta\sqrt{2y})\right)}dy + e^{-(\zeta/\delta)|x|}\right), \quad (41)$$

where Y_1 and J_1 are Bessel functions of first and second kind, respectively. Using asymptotics of these Bessel functions (see Abramowitz and Stegun [1968]), the denominator of the integrand of (41) is asymptotically equivalent to a constant when $y \to 0$ and to $y^{-\frac{1}{2}}$ when $y \to \infty$. Then $g(x) \sim \frac{1}{x^2}$ when $x \to 0$, and hence every path of this process has infinitely many jumps in each finite time interval[11]. Moreover, G satisfies all the assumptions of Theorem 2 in Eberlein and Jacod [1997], so the set of Equivalent Martingale Measures (EMM) denoted by $\mathcal{Q}$ is nonempty when the interest rate is constant.

Submitted on April 2000. Revised on April 2001.

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[11] See Breiman [1968], Ch.14.

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