A MODEL TO ESTIMATE THE US TERM STRUCTURE OF INTEREST RATES

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Resumo

A estrutura a termo da taxa de juros norte-americanas exerce um papel de muita importância na análise de ativos de renda fixa. Por exemplo, estimar precisamente a estrutura a termo norte-americana é um passo crucial para aqueles interessados em analisar os bônus Bradys brasileiros como IDUs, DCBs, FLIRBs, EIs, etc. Neste trabalho os autores apresentam um modelo estatístico para estimar a estrutura a termo das taxas de juros norte-americanas. Os autores abordam neste trabalho todas as questões de importância que os dirigiram no processo de implementar o modelo proposto, concentrando em pontos práticos de importância como eficiência computacional e robustez da implementação final, propriedades estatísticas do modelo final, etc. Exemplos numéricos são apresentados de forma a ilustrar o uso diário do modelo.

Abstract

The US term structure of interest rates plays a central role in fixed-income analysis. For example, estimating accurately the US term structure is a crucial step for those interested in analyzing Brazilian Brady bonds such as IDUs, DCBs, FLIRBs, EIs, etc. In this work we present a statistical model to estimate the US term structure of interest rates. We address in this report all major issues which drove us in the process of implementing the model developed, concentrating on important practical issues such as computational efficiency, robustness of the final implementation, the statistical properties of the final model, etc. Numerical examples are provided in order to illustrate the use of the model on a daily basis.

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1. Introduction.

The *spot interest rate* for a given maturity is defined as the yield of a pure discount bond [Fabozzi (1991)] with the same maturity. For example, if the current price of a discount bond is d, and the principal p is to be repaid t semesters ahead, the annualized spot interest rate r_t, can be obtained by solving the equation

$$\frac{d}{p} = \frac{1}{\left(1 + \frac{r_t}{2}\right)^t}.$$

The *term structure of interest rate* is a mapping which ralates the spot interest rate r_t to the term t [Fabozzi (1991); Fabozzi & Fong (1994)]. In particular, the US term structure of interest rates provides the spot interest rate prevalent in the US treasury market – covering Treasury Bills, Treasury Notes and Treasury Bonds – for t varying from one day to thirty years.

Estimating the US term structure of interest rates in one of the crucial steps required from any investor in the global fixed-income market. Consequently, investors interested in trading Brazilian Brady bonds must have available the US term structure of interest rates in order to price, trade, hedge and swap these bonds. From the equation given above we observe that the problem of obtaining the US term structure would be a very easy one if pure discount bonds were available for all maturities of interest for the investor. Unfortunately, this is not the case for maturities above one year in the US treasury market [Fabozzi (1991)].

In this work we propose a statistical model to estimate the US term structure of interest rates; in other words, we propose a statistical model which provides the value of r_t for $0 \leq t \leq 60$ (where we measure t in semesters).

As an example of the use of the US term structure let us consider the case of pricing a US Treasury Bond with exact twenty years for maturity, paying coupons equal to c every semester and a principal payment equal to p twenty years from now. The price of such a bond would be given by:

$$\text{price} = \sum_{\ell=1}^{40} \frac{c}{2\left(1+\frac{r_\ell}{2}\right)^{\ell}} + \frac{p}{\left(1+\frac{r_{40}}{2}\right)^{40}}.$$

As we observe from the expression above, the relation between the price of the bond and the spot interest rates prevalent in the market is a nonlinear one. If we define the *discount function* to be

$$\delta(t) = \frac{1}{\left(1+\frac{r_t}{2}\right)^{t}}, \qquad 0 \leq t \leq 60,$$

the price of the bond could be obtained as

$$\text{price} = \sum_{\ell=1}^{40} \frac{c}{2}\,\delta(\ell) + p\delta(40).$$

Two characteristics of the discount function which make it very important are:

1) It completely specifies the term structure of interest rates – and vice-versa. Thus, if we estimate the discount function we can easily obtain the term structure of interest rates, or what is equivalent, it suffices to estimate the discount function for the purpose of obtaining the term structure of interest rates.
2) The price of any option-free fixed-income instrument is linearly related to the discount function. This is a very important property of the discount function since it reduces the problem of

estimating the term structure from a nonlinear regression problem to a linear regression problem (plus solving several nonlinear equations which are very easy to solve).

There are other possibilities that completely specify the term structure of interest rates, such as the *forward rates* [Fabozzi (1991)]. Using forward rates to estimate the term structure of interest rates will require nonlinear estimation methods, however, which we decided to avoid for reasons of computational efficiency and robustness of the final model. The literature on the estimation of the US term structure of interest rates is vast at this point in time; see for example Bliss (1994), Buono, Gregory-Allen & Yaari (1992), Coleman, Fisher & Ibbotson (1992), Diament (1993), Fisher, Nychka & Zervos (1994), Litterman, Scheikman & Weiss (1991), McCulloch (1975b), and Vasicek & Fong (1982). Estimating other G7 term structure is of crucial importance for global fixed-income investors, as Barone, Cuoco & Zautik (1991), Kikugawa & Singleton (1994) and Schaefer (1981) illustrate in the case of Italy, Japan and the United Kingdom respectively. In this work we concentrate on the estimation of the US term structure of interest rates.

In terms of its organization, in the next section we present the statistical model developed by us. Following that, we discuss several topics which are important when implementing the proposed model, illustrating its use on a daily basis. Future directions for research are provided in the conclusion of the work.

2. The Model.

Two guidelines drove us when developing a model to estimate the US term structure of interest rates:

1) The estimated model should fit as well as possible the quoted prices of US Treasury Bills, Notes and Bonds.
2) Due to arbitrage opportunities, the estimated term structure could not deviate much from the market-observed STRIPS curve

[Fabozzi (1991)]; see Daves & Ehrhardt (1993) and Lim & Livingston (1995) for two interesting discussions on the subject.

In order to capture both objectives in a sole model we relied on the use of an emerging field of operations research: multiobjective programming.

We present in the remainder of this section the model developed by us, introducing simultaneously the notation used throughout the work:

1) In view of what has already been said before, our model concentrates in estimating the discount function $\delta(\cdot)$. When modeling the discount function we relied on exponential splines as originally done in McCulloch (1975), and later reused in Fisher, Nychka & Zervos (1994) and Vasicek & Fong (1982). We model the discount function as

$$\delta(t) = \sum_{k=1}^{N_1-1} \sum_{\ell=0}^{C_k} \beta_{\ell k} e^{-\ell \alpha t} I_{(knot_k \leq t < knot_{k+1})},$$

where $0 = knot_1 < knot_2 < \cdots < knot_{N_1} = 60$ are the knots chosen for the fit, C_k denotes the number of powers of $\{e^{-\ell \alpha t}\}_{1 \leq \ell \leq C_K}$ used between $knot_k$ and $knot_{k+1}$, α is the limit of the forward rates[1], $I_{(\cdot)}$ denotes the indicator function[2], and $\beta_{\ell k}$ are unknown linear coefficients.

2) Several constraints were imposed on the estimated discount function:

- $\delta(0) = 1$.
- $\delta(t_{T-Bill}) = \frac{1}{\left(1+\frac{yield(t_{T-Bill})}{2}\right)^{t_{T-Bill}}}$, where t_{T-Bill} is related to the US Treasury Bill with the longest maturity available for

[1] That is, $\lim_{t \to \infty} f(t) = \alpha$, where $f(\cdot)$ denotes the instantaneous forward rate ([8]).

[2] That is, $I_{(knot_k \leq t < knot_{k+1})} = 1$ if $knot_k \leq t < knot_{k+1}$, and 0 otherwise.

negotiation in the secondary market, and $yield(t_{T-Bill})$ denotes its bond-equivalent yield.

- Continuity is required from the discount function and its first derivative at $knot_2, knot_3, \cdots, knot_{N_1-1}$.

3) The distance between the discount function and the STRIPS curve is measured by

$$\delta(s_j) = \theta_{s_j} + \frac{1}{\left(1 + \frac{strip(s_t)}{2}\right)^{s_t}} \qquad \forall j = 1, 2, \cdots, N_2,$$

where N_2 is the number of STRIPS used, s_j is the maturity of the j^{th} STRIPS used and $strip\ (s_j)$ its bond-equivalent yield.

4) The residuals of the statistical fit obtained for the market-observed prices are given by

$$p_j + a_j + \frac{c_j d_j}{2}\,\phi = \sum_{i=1}^{\lfloor t_j \rfloor} \frac{C_j}{2}\,\delta(t_j - i) + \delta(t_j) + \varepsilon_j, \quad \forall j = 1, 2, \cdots, N_3,$$

where N_3 denotes the number of Treasury Bills, Notes and Bonds used, p_j the mean of the bid and ask prices for the j^{th} debt instrument under consideration, a_j its accured price, d_j its duration, ϕ a tax-related variable [Litzenberger & Rolfo (1984); McCulloch (1975b); Prisman (1990)] and ε_j the residual of its fitted price.

5) Sign restrictions were imposed on some variables, such as

$$0 \leq \delta(t) \leq 1, \qquad 0 \leq t \leq 60.$$

The free variables were

$$-\infty < \phi, \varepsilon_1, \cdots, \varepsilon_{N_3}, \theta_{s_1}, \cdots, \theta_{s_{N_2}}, \beta_{01}, \cdots, \beta_{C_{N_1-1}N_1} < \infty.$$

6) Finally, the objective function – to be minimized – is given by

$$\sum_{j=1}^{N_3} \frac{|\varepsilon_j|^g}{\left(1+(p_j d_j-1)\, I_{(d_j\geq 2)}\right)^{\frac{g}{2}}} + \rho \sum_{\ell=1}^{N_2} \left(|\phi_{s_\ell}| + (\nu-1)\theta^-_{s_\ell}\right),$$

where g, ρ, ν are user-defined parameters which should be chosen with the computational efficiency and robustness of the final implementation in mind. For example, if the user sets $\rho = 0$ he will discard the STRIPS curve from his fit; on the other hand, if the user sets ρ to a very large number he obtains a fit which follows the STRIPS curve very closely. Another example is obtained by setting $g = 1$, $\rho = 0$, which is a combination that results in the L_1 estimation of the term structure; by setting $g = 2$, $\rho = 0$ we would obtain a least squares estimation of the terms structure. The choice of the parameter ν is related to the liquidity [Lim & Livigston (1995); McCulloch (1975a)] stripping and reconstitution [Daves & Ehrhardt (1993)] of STRIPS, US Treasury Notes and Bonds.

Choosing the parameters for the model is a user-dependent problem which we do not consider simple. It requires an understanding of both the model and the current conditions prevalent in global fixed-income markets. We believe that g is the easiest parameter to choose for the following two reasons:

1) If $g = 1$ the model above can be formulated as a linear program, which results in a more efficient computational implementation, as discussed below.
2) If $g = 1$ the model relies on L_1 estimation, which is a robust estimator, contrary to the least squares estimator. Gross errors in the data are less influential on the final estimated coefficients when $g = 1$, this proving to be an important improvement over previous suggestions as we observed from our daily experience.

In the next section we address some points related to the implementation of the model described above.

3. Operational Details.

The two most important point which drove us during the final implementation of the model where:

1) Computational Efficiency. It is always important to remember that information should be analyzed and provided to traders and portfolio managers and quickly as possible in financial markets. To that end, requiring the final implementation of our model to be computationally efficient is a necessity for it to be fully used by the traders and portfolio managers of an institution. The model described before was implemented in order to attain the maximum computational efficiency possible, allowing thus incoming news to be processed and passed to decision markers as quickly as possible.
2) Robustness of the model. The models used in the finance industry must always be able to withstand the presence of gross errors in the data inputted; if this is not the case, these models will not be trusted by traders and portfolio managers, and will consequently be left unused. Resisting the presence of abnormal data is important because when mathematical models are run with live information, gross errors cannot be allowed to propagate into the decision making process through those models under use.

The robustness of our model was achieved through the use of constraints (as described before) and by choosing $g = 1$. We concentrate in remainder of this section only on the computational efficiency of our final implementation.

In Figure 1 we present an overview on the operational details used. We can divide the process outlined in Figure 1 basically in four steps:

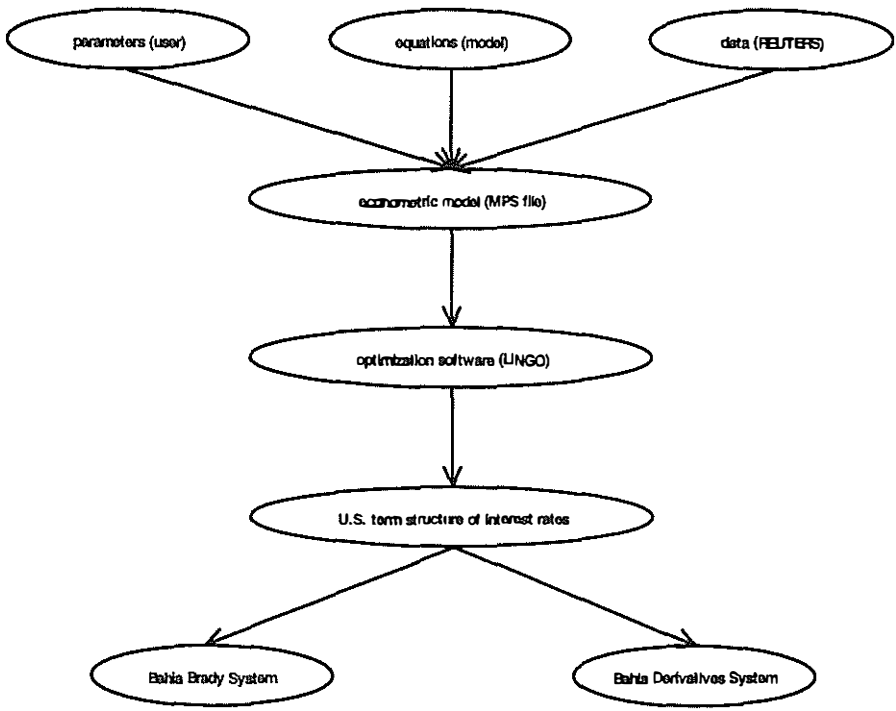


Fig. 1: An overview of the implementation

1) The parameters and the quotations of US Treasury Bills, Treasury Notes, Treasury Bonds and STRIPS are obtained from the user and REUTERS. This information is then passed into the equations of the model.

2) The econometric model is generated as a MPS file[3] in order to attain maximum computational efficiency in the next step.

[3] A MPS file is a type of input file frequently used in large-scale optimization which allows a very fast preprocessing of information by optimization softwares, as well as a very efficient generation of data structures in optimization problems.

3) The econometric model is fed into an optimization sofware to obtain the final estimate of the term structure.
4) With the estimated coefficients obtained, the discount function is generated and made available for other softwares developed internally in Banco da Bahia (such as presented in Gonçalves & Issler (1995) in the case of the Bahia Derivatives System).

In Banco da Bahia the US term structure of interest rates is computed from scratch every ten minutes in order to keep up with the latest quotations in the US treasury market. The updated US term structure is made available for other systems which may recall the latest estimated curve to price, hedge, estimate the implied volatility and swap Brady bonds, as well as to price options on these instruments.

More operational details can be found in Duarte Jr. & Soares (1995), or requested directly from the first author.

4. Numerical Results.

For the purpose of illustrating the use of the proposed model, we present the estimated US term structure of interest rates on 23 August 1995[4]. In Figure 2 we depict the fitted and the STRIPS curve so that the reader can compare them. We can clearly observe that the shape of both curves do not differ much for maturities above ten years, this not being the case however for short term maturities: let us notice the presence of "outliers" in the STRIPS curve up to five years, illustrating our concerns with the presence of gross errors in the data inputted into the model. It is clear that the fitted term structure resisted very well the presence of these bad data, as

[4] Termstructure using $knot_1 = 0, knot_2 = 0.9233, knot_3 = 10, knot_4 = 20, knot_5 = 30, g = 1, \rho = 10$ and $\nu = 1$. Obtained using REUTERS prices for STRIPS, US Treasury Bills, Treasury Notes and Treasury Bonds as of 5:00pm Rio de Janeiro time.

it should in fact be the case in view of all precautions we had when developing the model. For the sake of completeness we provide in Figure 3 the fitted discount function.

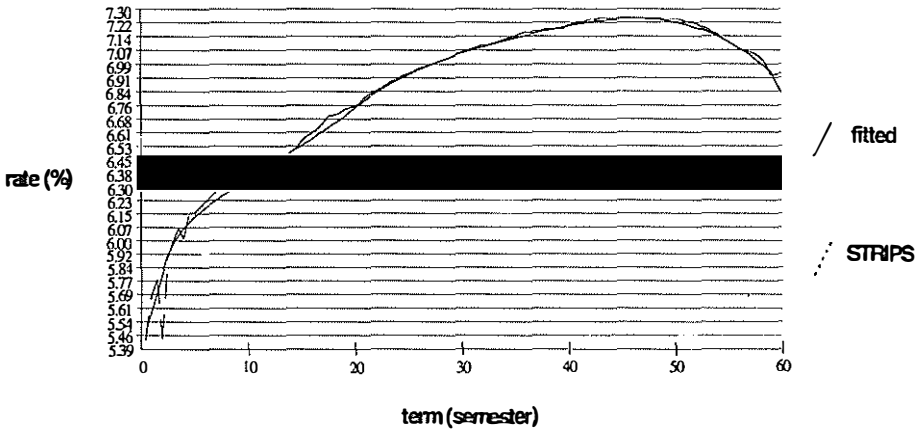


Fig. 2: US term structure of interest rates on 23/08/1995

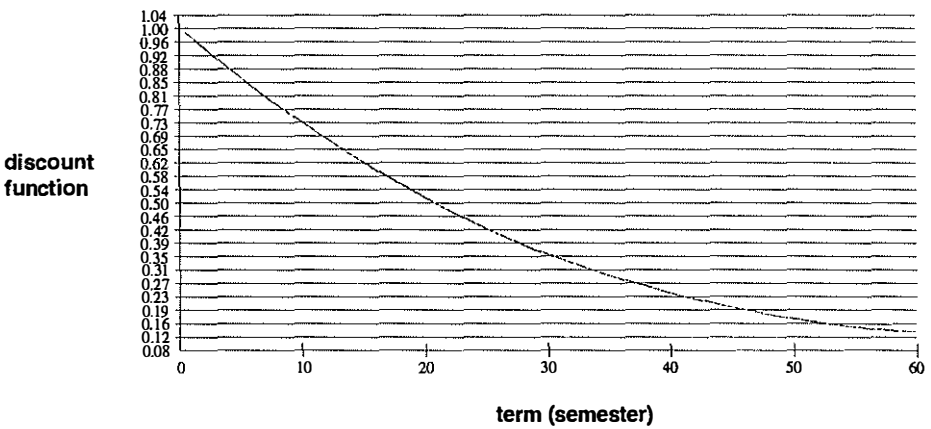


Fig. 3: Discount function on 23/08/1995

In Figure 4 we provide the fitted and market-observed REUTERS prices in a longitudinal plot. We clearly observe that both series display a very similar behavior, implying that the model provided a very good fit for the market-observed REUTERS prices. An important measure of the quality of our fit is the average relative error[5], which in this case was 0.392%.

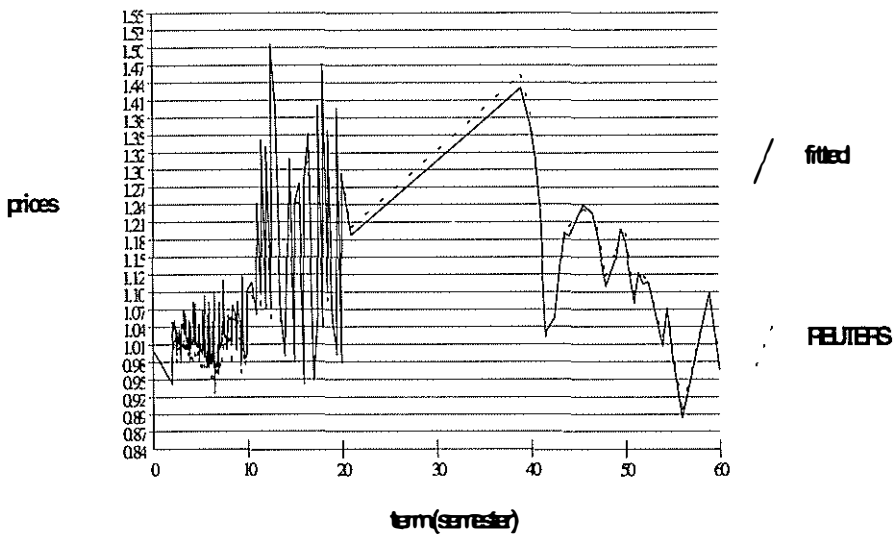


Fig. 4: Fitted and REUTERS prices on 23/08/1995

In Table 1, Table 2 and Table 3 we illustrate the impact of modifying parameters in the model (such as ν, ρ, N_1, etc.) and their effects on the average relative error observed. From our experience with the model presented we can safely say that the most important parameters to calibrate the model are the knots and the ρ chosen.

[5] If we denote the fitted prices by $\hat{p}_1, \hat{p}_2, \cdots, \hat{p}_n$, and the market-observed REUTERS prices by $p_1, p_2, \cdots, p_n$, the average relative error is given by $\frac{1}{n} \sum_{i=1}^{n} \frac{|p_i - \hat{p}_i|}{p_i}$.

Knowing the US treasury market and developing a feeling on how the model behaves when parameters are changed are crucial steps when obtaining a US term structure of interest rate with the model proposed in here.

Table 1

Average relative error for different parameters *

Average Relative Error			
ρ	$\nu = 1.0$	$\nu = 1.5$	$\nu = 2.0$
0.01	0.549%	0.568%	0.573%
0.1	0.510%	0.508%	0.603%
1	0.529%	0.534%	0.631%
10	0.500%	0.442%	0.528%
100	0.518%	0.460%	0.543%
1000	0.523%	0.486%	0.563%

* With $N_1 = 9$, $knot_1 = 0.0$, $knot_2 = 0.94248$, $knot_3 = 2.0$, $knot_4 = 4.0$, $knot_5 = 6.0$, $knot_6 = 8.0$, $knot_7 = 10.0$, $knot_8 = 20$, $knot_9 = 30.0$ and $\alpha = 0.034285$. Results obtained on 21 August 1995 at approximately 5:00pm Rio de Janeiro time.

Table 2

Average relative error with and without the tax parameters ϕ *

ρ	**Average Relative Error** when $\phi = 0$	**Average Relative Error** when ϕ is a free variable
0.01	0.583%	0.568%
0.1	0.878%	0.508%
1	1.284%	0.534%
10	2.381%	0.442%
100	2.217%	0.460%
1000	2.235%	0.486%

* The same knots and α as in Table 1, but with $\nu = 1.5$.

Table 3

Average relative error for different knots *

N_1	**Average Relative Error**	**Knots**
4	0.582%	0, 0.92498, 10, 30
5	0.450%	0, 0.92498, 10, 20, 30
6	0.439%	0, 0.92498, 5.5, 10, 20, 30
7	0.439%	0, 0.92498, 3, 6.5, 10, 20, 30
8	0.447%	0, 0.92498, 2, 4, 7, 10, 20, 30
9	0.442%	0, 0.92498, 2, 4, 6, 8, 10, 20, 30

* The same data as in Table 2, but with $\rho = 10$.

5. Conclusion.

In this paper we presented a model for estimating the US term structure of interest rates. This model is currently being used on a daily basis by the first author for pricing, hedging, trading and swapping Brazilian Brady bonds, as well as option on these instruments.

In this work we showed and commented on the motivation behind the development and implementation of the model proposed. Computational efficiency and the robustness of the final implementation where critical issues when choosing the parameters, inputing the data and generating the equations of the model. So far we have not experienced any difficulties with the operational side of the proposed model.

Numerical results were also reported to illustrate how the parameters and the data influence the estimated term structure. We stress the fact that using the proposed model required from us tuning parameters until we felt confident with the output, as well as observing the US treasury market on a daily basis in order to follow closely the market expectations for US Treasury instruments.

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