An SDF Approach to Hedge Funds' Tail Risk: Evidence from Brazilian Funds[*]

Laura Simonsen Leal[**]
Caio Almeida[***]

Abstract

The main purpose of this paper is to propose a methodology to obtain a hedge fund tail risk measure. Our measure builds on the methodologies proposed by Almeida and Garcia (2015) and Almeida et al. (2016), which rely in solving dual minimization problems of Cressie Read discrepancy functions in spaces of probability measures. Due to the recently documented robustness of the Hellinger estimator, we adopt within the Cressie Read family, this specific discrepancy as loss function. From this choice, we derive a minimum Hellinger risk-neutral measure that correctly prices an observed panel of hedge fund returns. The estimated risk-neutral measure is used to construct our tail risk measure by pricing synthetic out-of-the-money put options on hedge fund returns of ten specific categories. We provide a detailed description of our methodology, extract the aggregate Tail risk hedge fund factor for Brazilian funds, and as a by product, a set of individual tail risk factors for each specific hedge fund category.

Keywords: Asset Pricing; Stochastic Discount Factor; Risk-Neutral Probability; Tail Risk; Hedge Funds.

JEL Codes: G12, G13, G17.

Laura Simonsen Leal and Caio Almeida

1. Introduction

Investors dread the possibility that their portfolios will face extreme downturns. Therefore, it is only natural that they would search for ways of identifying and hedging against potential tail risks. What we endeavor to present in this paper is the first step for an identification of such potential risks. We suggest a new approach to measure hedge fund tail risks. We adopt the methodology proposed by Almeida et al. (2016) (AAGV). However, instead of working with returns from the equity market, we rely on the convergence trading literature and use returns of hedge funds in our estimation, as proposed by Fernandes and Santos (2015).

Given observed returns from a set of basis assets, Hansen and Jagannathan (HJ, 1991) introduce a methodology to estimate minimum variance stochastic discount factor bounds. The methodology offers, as a by-product, implied Stochastic Discount Factors (SDFs) that are linear functions of these observed returns. Since then, their methodology has been applied in many empirical analysis including but not limited to diagnostic of asset pricing models, portfolio management, predictability issues, and performance analysis of mutual and hedge funds (see Ferson, 2003).

Almeida and Garcia (2015) generalize the quadratic loss function in HJ to a whole family of convex functions related to the Cressie Read family of discrepancies. As a by-product, they obtain a corresponding family of implied admissible SDFs that are hyperbolic functions of observed returns.

This family of implied SDFs, which contains as a particular case the linear HJ SDF, has been adopted in Almeida and Garcia (2013) to calculate ranges of performance for international hedge funds. Since each given admissible SDF presents a different structure for risk compensation in different states of nature, Almeida and Garcia (2013) show that performance is SDF dependent. That is, depending on each specific hedge fund class (i.e., the structure of its returns), it is possible that a whole range of different values for its performance is obtained. More specifically, for some particular hedge fund classes, performance may consistently vary across different members of the Cressie Read family. This variability is related to how much weight each implied SDF puts in higher order moments (like skewness and kurtosis) of observed returns of the risk factors being priced as basis assets.

Since each strictly positive admissible SDF can be associated with a risk-neutral measure (see Duffie, 2001), implied Cressie Read SDFs provide a way of risk-neutralizing measures, objects, and/or moments. This was precisely the insight pursued by AAGV, who adapted the methodology proposed by Almeida and Garcia (2015) to estimate a tail risk factor extracted from portfolios of U.S. stocks. AAGV show that this tail risk factor is priced in the cross-section of individual stocks in the U.S., and is able to forecast many important U.S. macroeconomic variables.

Kelly and Jiang (2014) measure tail risk from a cross-section of stock returns adopting a dynamic power law structure for the shape of the tails. It is interesting

to note that their tail risk factor has a strong predictive power for aggregate market returns. Building on this work, Fernandes and Santos (2015) use the same power-law structure to extract a tail risk factor from a cross-section of U.S. hedge fund returns. The rationale behind the idea of Santos relies on the convergence trading literature. According to Stein (2009), the fact that some hedge funds follow similar strategies and market signals creates negative externalities between them. Since the arbitrageur cannot know how many others are taking the same strategies and positions (overcrowding), leveraged positions can transform an idiosyncratic shock into a systematic shock in the whole industry. The shock spreads to the industry through fire sales that inflicts losses in the funds with the same position, causing other rounds of sales from the leveraged funds.[1] In this way, Santos hedge fund tail risk factor has clearly stronger predictive ability to forecast future returns for different hedge fund categories, than the original tail risk factor extracted by Kelly and Jiang.

We intended to explore the richness of the Cressie Read family of implied SDFs to extract a hedge fund tail risk factor. Naturally, such factor could have been extracted from a set of risky factors observed on primary markets (equities, currencies, interest rates, options, etc...) as done in Almeida and Garcia (2013) (or in Kelly and Jiang (2014), who only adopt stocks). However, the approach proposed by Fernandes and Santos (2015) in adopting hedge fund returns to obtain a tail risk factor, inspired a different path for the use of Cressie Read SDFs within the hedge fund industry.

The contribution of this paper, although specific is made clear here. We work at the intersection of the research proposed in Almeida and Garcia (2015), Almeida and Garcia (2013) and AAGV (2015) to propose a way to measure Tail Risk for the Brazilian hedge fund industry. This factor is directly extracted from hedge fund returns as done in Fernandes and Santos (2015).

Our results show that the tail risk factor estimated from the Brazilian hedge fund returns presents weak correlation with the volatility of the equity market index Ibovespa. We find a correlation of only 0.14 between the two series. However, the HP filtered series present a correlation of 0.78, indicating that the tail risk move along with Ibovespa's volatility on a lower frequency.

We also estimate the tail risk of the 10 hedge fund styles. We find a considerable variability between the tail risks. Nevertheless, almost all tail risks are positively correlated with each other. As expected, categories more focused on capital hedging such as "Protected Capital", "LS-Neutral", and "Specific Strategy" present the lowest correlation with other tail risks. On the other way, "Balanced", "Macro", "Multistrategy" and "Fund of Funds" are the categories with the highest correlation. When comparing the correlation with the Hellinger's implied Ibovespa tail risk, the "Balanced" and "Protected Capital" strategies present the highest

---

[1]See Kondor (2009) for an example of a model with convergence tranding and endogenous price divergence.

correlation, 0.55 and 0.31, respectively. However, the aggregate hedge fund tail risk presents a correlation of only 0.09 with Ibovespa tail risk.

The remainder of this paper is as follows. Section 2.1 presents some adequate ways of measuring tail risk adopted in the literature. Section 2.2 presents a brief description of the hedge funds performance literature. In Section 3, our methodology is presented in details. In Section 4, we estimate both an aggregate tail risk measure as well as a tail risk measure for each individual hedge fund category, using hedge funds return data from ANBIMA.[2] In Section 5, we conclude, offering a short description of this paper's main contribution.

2. Literature Review

2.1 Tail risk

Tail risk is the probability of extremely large losses in portfolio returns. Because investors are tail risk averse, there is a positive relation between tail risk and future returns. Naturally, the return required by investors to hold assets increases when tail risk increases.

One of the first researchers to describe this relation between tail risks and asset prices was Rietz (1988), who extends the Mehra and Prescott (1985) model to include a rare disaster state. With this model, he hoped to solve the equity premium puzzle. Critics, however, claimed that his estimates were not compatible with historical U.S. consumption growth. Barro (2006) later extended his model to international data and found that this new calibration finds reasonable estimates for the equity premium.

In recent literature, time series analysis has given way to cross-section approaches of tail risk estimation. Bollerslev and Todorov (2011) use high-frequency data for S&P 500 futures and closing bid and ask quotes for S&P 500 options to estimate a model-free index of investor's fear. Their empirical findings suggest that historically large equity and variance risk premia are probably compensation for tail events. They, however, have a different definition of tail risk from the one stated above. For them, tail risk is seen as daily jumps in asset prices, which are usually not as large.

Siriwardane (2013) constructs a disaster risk measure based on option prices. It consists of subtracting an OTM call from and OTM symmetrical put, normalized by the firm's current stock price. His measure proxies for the ex-ante disaster risk of a firm's stock. Firms that present higher disaster risk should require higher equity returns in equilibrium, if this risk is priced in. Indeed, they find that a zero-cost equity portfolio that is exposed to high disaster risk stocks earns excess annualized returns of 12.13%, even after controlling for standard factors such as Fama-French, momentum, liquidity and volatility risk.

---

[2] We thank very much both the technical and the data teams at ANBIMA for providing access to their private dataset on hedge fund returns.

Kelly and Jiang (2014) measure tail risk using the cross-section of returns. They use the fact that some assets are probably going to go through tail events in a given month to create a global measure of common fluctuations in tail risk. Their tail risk measure results from a dynamic power law structure that considers two parameters, one representing the systematic tail risk and the other representing asset-specific tail risk. They show that tail risk has strong predictive power for aggregate market returns.

Almeida et al. (2016) propose a novel way to estimate tail risk incorporating risk-neutral information using a panel of stock returns. This approach is very interesting because it precludes the use of options data, which are not available, or do not have enough liquidity, in many markets. In their paper, they use 25 Fama-French size and book to market portfolios to estimate tail risk and forecast aggregate U.S. macroeconomic activity indexes and yield spreads.

In short, here are three ways to measure tail risk:

(i) Using option prices, as in Bollerslev and Todorov (2011) and Siriwardane (2013);

(ii) Using the cross-section of returns, as in Kelly and Jiang (2014);

(iii) Using the SDF approach and a synthetic put to estimate tail risk, as in Almeida et al. (2016).

2.2 Hedge fund performance

In normal times, hedge funds provide liquidity and increase the efficiency of the financial markets. In times of crisis, on the other hand, they might contribute to increased market volatility and downfalls. It is important to keep in mind that hedge funds can be a great source of financial risk to a country's economy. They face liquidity risks, volatility risks, investor protection and same strategy risks, which happen when hedge funds look at the same macroeconomic scenario, build the same strategies and get squeezed by the lack of counterparts to the operation. The extent to which hedge funds are exposed to tail risk can have a great impact on the overall economy and, therefore, it is very important to have a measure that helps us understand this exposure. There have been some very interesting contributions in the literature in this regard.

Using their power law methodology, Kelly and Jiang (2012) find that tail risks are the main drivers of hedge fund returns in both cross-section and time series. This indicates that a large fraction of hedge fund returns can be interpreted as compensation for providing insurance against disasters. That is, whether a fund is protected or not against tail risks directly influences hedge fund returns.

Building on the approach presented in Kelly and Jiang (2014) that uses Extreme Value Theory to approximate lower tail, Fernandes and Santos (2015) proposes a new factor to control for extreme downside events risk. Using information

embedded in a panel of hedge funds performance, they create a Hedge Fund Tail Risk measure (HFTR) and show that this new measure has greater forecasting explanatory power than the one presented by Kelly and Jiang (2014). Their approach can account for over-crowding and over-leverage effects, as well as for equity market risks.

Adrian et al. (2011) link hedge funds to the risk of systemic crisis and study interdependencies between different styles in times of distress. Using quantile regressions, they find that tail sensitivities between strategies are higher in difficult times, and identify seven factors that explain a large part of tail risks. They stress that offloading tail risks that derive from these seven factors might come at the cost of lower returns for individual funds.

Capocci and Hübner (2004) investigate hedge funds performance and persistence using a combination of various asset pricing models, including an extension of Carhart (1997), Fama and French (1998), Agarwal and Naik (2002) and an additional factor that takes investment in emerging market bonds into account. They find that only one fourth of hedge funds have significant positive excess returns and that over-performance is usually constant over time. However, they also find that the great majority of these funds suffered from the Asian crisis, which tells us that they are exposed to global tail risks.

The approach followed by Bailey et al. (2004) analyses hedge funds' performances by using stochastic discount factors. They use Hansen and Jagannathan (1997) distance measure within the SDF framework in a way that allows them to take into account the non-arbitrage requirement. Since hedge funds are able to trade derivatives and have dynamic strategies, they are often able to take advantage of negative SDFs. By imposing non-arbitrage and requiring the SDFs to be positive, they are able to take into account hedge fund trading strategies and non-linearity in returns and thus surpass linear asset pricing models.

In this paper, we build on the stochastic discount factor approach used by Bailey et al. (2004), but instead of using Hansen and Jagannathan (1997) distance measure, we use the generalization of their model for general convex functions as presented in Almeida and Garcia (2015). In the next section, we present their methodology, which relies on the solution of a dual problem and proceed to building our tail risk measure for hedge funds.

3. Methodology

3.1 Dual problem

Let $(\Omega, \mathbb{F}, P)$ be a probability space and let R denote a K-dimensional random vector on this space representing the returns of K primitive basis assets. In the static setting we are considering, an admissible SDF is a random variable m for which $\mathbb{E}[mR]$ is finite and the Euler equation $\mathbb{E}[mR] = 1_K$ is satisfied, where 1_K is a vector of ones with K dimensions.

The set of admissible SDFs will vary according to the market structure. Complete markets will give us an unique SDF and incomplete markets will give us an infinity of SDFs. We can restrict the latter to be positive if we impose the absence of arbitrage. This is exactly what we will do. Brazilian markets are incomplete, i.e., the number of states of nature (T) is larger than the number of basis assets, and in the absence of arbitrage we will get an infinity of positive stochastic discount factors. For each one of them there will be a corresponding risk neutral probability.

In their seminal paper, Hansen and Jagannathan (1991) minimize a quadratic function in the space of non-negative admissible stochastic discount factor with fixed mean a to find a minimum variance bound for the SDFs. They do so in the context of (1), where R_{t+1} is the gross return on time $t + 1$ and m_{t+1} is the stochastic discount factor. Assuming that the process (m_t, R_t) is sufficiently regular (e.g., stationary and ergodic) such that a time series version of the law of large number applies, sample moments will converge to population counterparts as the sample size T becomes large.

$$1 = E[m_{t+1}R_{t+1}]$$
$$m_{t+1} = f(data, parameters) \tag{1}$$

Now, given a discrepancy function ϕ, Almeida and Garcia (2015) find that the (in sample) generalized minimum discrepancy problem can be stated as:

$$
\begin{aligned}
\hat{m} = \underset{m_1,\ldots,m_T}{\operatorname{argmin}} \quad & \frac{1}{T}\sum_{i=1}^{T}\phi(m_i) \\
\text{subject to} \quad & \frac{1}{T}\sum_{i=1}^{T}m_i\left(R_i - \frac{1}{a}1_K\right) = 0_K \\
& \frac{1}{T}\sum_{i=1}^{T}m_i = a \\
& m_i > 0, \forall i
\end{aligned}
\tag{2}
$$

where the first and second restrictions are the sample equivalent to $\mathbb{E}[m(R - \frac{1}{a}1_K)] = 0_K$ and $\mathbb{E}[m] = a$, respectively, and the last restriction imposes positivity of the stochastic discount factor, which makes it compatible with the absence of arbitrage.

Furthermore, Almeida and Garcia (2015) make use of the results in Borwein and Lewis (1991) to prove that, in general, the optimization problem stated in (2) can be solved in a simpler finite dimensional dual space. The problem then becomes:

$$\hat{\lambda} = \underset{\alpha \in \mathbb{R}, \lambda \in \Lambda}{\operatorname{argsup}} \quad a * \alpha - \frac{1}{T} \sum_{i=1}^{T} \phi^{*,+} \left(\alpha + \lambda' \left(R_i - \frac{1}{a} 1_K \right) \right) \tag{3}$$

where $\Lambda \subseteq \mathbb{R}^K$, λ is a vector of K Lagrange multipliers that comes from the Euler equations for the primitive basis assets, α is the Lagrange multiplier that comes from the original restriction $\mathbb{E}[m] = a$ and $\phi^{*,+}$ is the convex conjugate of ϕ restricted to the positive real line.

$$\phi^{*,+} = \underset{w>0}{\sup} \quad zw - \phi(w) \tag{4}$$

Assuming that the discrepancy ϕ above belongs to the Cressie and Read (1984) discrepancy family, i.e.,

$$\phi^{\gamma}(m) = \frac{m^{\gamma+1} - a^{\gamma+1}}{\gamma(\gamma+1)} \tag{5}$$

with $\gamma \in \mathbb{R}$. The use of this family of discrepancy functions allows for the generalization of several particular restrictions on the space of SDFs, such as those derived by Hansen and Jagannathan (1991), Snow (1991), Stutzer (1995), Bansal and Lehman (1997) and Cerny (2003).

Considering $\gamma < 0$, we arrive at closed form formulas for λ and for the empirical estimates of minimum discrepancy SDFs ($\hat{m}_{MD}$):

$$\hat{\lambda} = \underset{\lambda \in \Lambda_{CR}}{\operatorname{argsup}} \quad \frac{1}{T} \sum_{i=1}^{T} \left(\frac{a^{\gamma+1}}{\gamma+1} - \frac{1}{\gamma+1} \left(a^{\gamma} + \gamma \lambda' \left(R_i - \frac{1}{a} 1_K \right) \right)^{\frac{\gamma+1}{\gamma}} \right) \tag{6}$$

where $\Lambda_{CR} = \lambda \in \mathbb{R}^K \mid \forall i = 1, \ldots, T; (a^{\gamma} + \gamma \lambda'(R_i - \frac{1}{a} 1_K)) > 0$ and

$$\hat{m}_i^{MD} = a \frac{\left(a^{\gamma} + \gamma \hat{\lambda}_{\gamma}' \left(R_i - \frac{1}{a} 1_K \right) \right)^{\frac{1}{\gamma}}}{\frac{1}{T} \sum_{i=1}^{T} \left(a^{\gamma} + \gamma \hat{\lambda}_{\gamma}' \left(R_i - \frac{1}{a} 1_K \right) \right)^{\frac{1}{\gamma}}} \tag{7}$$

3.1.1 Choice of Gamma

In this paper, we will work with $\gamma = -\frac{1}{2}$, following the robustness results of Kitamura et al. (2013). Their main theoretical result shows that the minimum Hellinger distance estimator (MHDE) possesses optimal min-max robust properties and that it remains semi-parametrically efficient when the model assumptions hold. Therefore, we choose the Hellinger estimator because it gives us higher chances of obtaining a tail risk factor that will be robust to the choice of the SDF used to calculate synthetic put prices.

3.2 Estimation of the risk neutral distribution

To estimate the risk neutral distribution, we need both the state probabilities and the stochastic discount factor. The SDF can be estimated for $\gamma = -\frac{1}{2}$. For the state probabilities, we will follow an approach that is less than perfect, but is traditionally used in finance, which assumes that each state of nature can occur with probability $\frac{1}{T}$.

This assumption allows us to compute the risk neutral probability quite easily

$$\pi_i^{RN} = \frac{m_i^{MD}(1 + r_f)}{T} \tag{8}$$

where $\pi(s) = \frac{1}{T}$, $\mathbb{E}[m] = \frac{1}{R_f} = \frac{1}{1+r_f}$ and $m(s)$ becomes the stochastic discount factor estimated in (7), with $\gamma = -\frac{1}{2}$.

These risk-neutral probabilities will allow us to estimate our tail risk measure.

3.3 Tail risk measure

To estimate a tail risk measure, we rely on the fact that prices usually reflect future expectations, i.e., what investors think about what might happen in future states. In particular, prices for out-of-the-money puts reveal information about the negative tail of the distribution of returns.

Therefore, it would be easier to use options data as a way to measure investors' perceptions. Unfortunately, in Brazil, we do not have enough options data available to estimate tail risks. In particular, out-of-the-money options are not very liquid when they exist. Thus, a way to get around this is to follow AAGV (2015) in constructing a tail risk measure that will be the average price of out-of-the-money synthetic puts for a pre-determined set of basis assets returns. This measure is obtained using the previously estimated risk-neutral probabilities.

First, we estimate the price of a synthetic put for each of the subcategories of hedge funds:

$$P = \sum_{i=1}^{T} \pi_i^{RN}[max(K - R_i, 0)] \tag{9}$$

where π_i^{RN} are the risk-neutral probabilities of each possible state, K is the (risk-neutral) strike price, which is the 10th percentile of the risk-neutral distribution, and $max(K - R_i, 0)$ is the payoff of the synthetic put in each possible state.[3]

The Tail Risk measure is the average of the synthetic puts that we have just obtained.

---

[3]AAGV (2015) provide a long robustness section, showing, among other things, that the 10th percentile has the most adequate sensitivity to derive a tail risk measure whenever the estimated SDFs present a small number of states, which is the case that we have in this paper with 30 states of nature.

3.4 Empirical Aspects of the Methodology

Once the methodology has been set, we approach the data with care. Our dataset comprises the daily return of over six thousand hedge funds during 1565 days. The data was provided by ANBIMA.[4]

Therefore, we felt the need to aggregate the returns in some way. We chose to aggregate them into the ten subcategories of hedge funds defined by ANBIMA. This helped us avoid the problem of over-specification generated by having more types of funds than data occurrences for each individual fund.

In this paper, tail risk is not measured as standard-deviations or as the probability of a huge historical loss. Instead, as defined above in Section 2, it depends on the average price of synthetic puts (on which we have defined the strike as a return quantile) that are generated on an thirty-day window. This approach implies that every month some subcategory of hedge funds will experience tail events and we will be capturing information related to the aggregate level of tail risk within a thirty-day moving window.

With this in mind, we transform the return panel in a $30 \times 10 \times 1565$ matrix representing 1565 moving windows of thirty days for the 10 types of hedge fund styles. On each of these windows, we calculated the ten Lagrange multipliers and, subsequently, the thirty entries for the Hellinger implied SDF using the methodology described before. The implied SDF will allow us to obtain the discrete risk-neutral probability given by a thirty-dimensional vector containing probabilities for each of the thirty scenarios within each moving window.

Within each moving window, to obtain the strike of the synthetic put priced for each of the ten individual hedge fund categories (or types), we extract the first decile of each class returns under the Hellinger-implied risk-neutral probability measure. The price of the synthetic put is obtained for each of the ten categories of hedge funds as being the expected value of the payoffs of our synthetic puts, under the Hellinger implied risk-neutral measure. Finally, our tail risk measure is obtained as the average of the prices obtained for the ten synthetic puts.

On a preliminary study using this new methodology, we had followed previous literature and used a fixed value for the interest rate throughout the analyzed period. This would have worked perfectly in current European or American markets due to their very low interest rate values, however, it is a strong generalization when using Brazilian interest rates data.

Interest rates in Brazil have changed considerably between 2009 and 2015, ranging from 7.12 in 01/16/2013 to 14.15 at the end of our sample. In our study, we have noticed that our tail risk measure is very sensitive to interest rate changes. Therefore, we use the whole time series for interest rates when calculating the tail risk measure. For each moving window (as defined above) we use the daily value

---

[4]Brazilian Association of Financial and Capital Market Institutions or, in Portuguese, "Associação Brasileira das Entidades dos Mercados Financeiros e de Capitais".

for the interest rate that relates to the first day of the window.

Using this methodology, we are sure to find a tail risk measure that takes into account the contemporary interest rate value, instead of depending on a single unconditional expected value that could potentially bias our results.

4. Empirical Performance

4.1 Data description

Our data for hedge funds was provided by ANBIMA.[5] According to their classification, Brazilian hedge funds are called multi-market funds. These funds have investment policies that involve several risk factors. They may invest in fixed income, exchange markets, stocks, etc. In this paper, we will use data for multi-market funds to measure tail risk.

Multimarket funds are subdivided in ten categories: Balanced, Protected Capital, Long and Short – Neutral, Long and Short – Directional, Macro, Trading, Multi-strategy, Funds of Funds, Interest and Currency and Specific Strategy.[6] The characteristics of these types of funds can be found in Appendix A and the daily return time series for each individual subcategory of hedge funds can be found in Appendix B.

To give the reader a better idea of the data that we are analyzing, we present a table with summary statistics.

Table 1
Summary statistics of daily hedge fund returns

Type of Fund	Sharpe	Mean	Median	Std Dev	Skew	Kurt	Min	Max
Balanced	0.2006	0.0374	0.0393	0.1839	-0.2711	5.1903	-0.9104	0.8230
Protected Capital	0.1104	0.0238	0.0312	0.2113	-0.2145	4.3465	-0.9738	0.7938
Long/Short-Neutral	0.4587	0.0434	0.0399	0.0934	0.1197	7.5071	-0.5772	0.6386
Long/Short-Directional	0.4173	0.0412	0.0374	0.0974	-0.0577	4.4684	-0.3628	0.5068
Macro	0.2494	0.0487	0.0470	0.1931	0.0644	6.9432	-0.9442	0.9930
Trading	0.1498	0.0324	0.0420	0.2125	-0.4425	7.7800	-0.9637	0.9401
Multistrategy	0.2619	0.0430	0.0410	0.1621	0.0488	5.8150	-0.7131	0.8787
Funds of Funds	0.4475	0.0390	0.0402	0.0860	-0.4288	8.3538	-0.6609	0.5278
Interest and Currency	0.6699	0.0386	0.0434	0.0568	-1.2477	15.6700	-0.4505	0.3825
Specific Strategy	0.3270	0.0446	0.0451	0.1347	-0.5301	8.5035	-0.9520	0.7338

This table reports summary statistics for the ten hedge fund style daily returns. For each type of fund we have 1594 observations. The Sharpe ratio is the ratio of mean excess returns to the standard deviation of returns, where we have used the daily equivalent of the 14.15% risk-free annual rate to obtain the mean excess returns, considering the year as having 252 working days.

As we can see from the summary statistics, Interest and Currency is the subcategory with the highest Sharpe ratio, even though the mean returns are not

---

[5]Brazilian Association of Financial and Capital Market Institutions or, in Portuguese, "Associação Brasileira das Entidades dos Mercados Financeiros e de Capitais"

[6]Their respective codes in ANBIMA's database are: 207, 208, 327, 328, 329, 330, 331, 332, 333 and 334.

as high as in other subcategories. This is due to the fact that it has the lowest standard deviation. Readers familiar with the hedge fund literature might recall that the Sharpe ratio is not the best way to evaluate hedge funds. This is due to the fact that they are able to leverage their portfolios and, by means of buying and short-selling assets, they are able to mask the value of their Sharpe ratio. Therefore, even though the Interest and Currency subcategory might look very interesting as an investment, it is important to check its exposure to tail risk before including it in a portfolio, as an investor. We will see later that, indeed, Interest and Currency not only presents the highest Sharpe ratio but also is the subcategory that presents smallest correlation (i.e., 30%) with our aggregate measure of Tail Risk.

From the summary statistics, we can also see that most of the funds have negative or very low skewness. We also find that all of them have positive, sometimes high, kurtosis, which indicates that the return distribution has fat tails when compared to the normal distribution. We test the data for normality using Royston's Multivariate Normality Test,[7] as in Adrian et al. (2011). According to Royston's test, the Brazilian hedge fund returns do not appear to follow a multivariate normal distribution (Royston's statistic: 999.8, p-value ¡ 0.001).

Therefore, we have checked that, indeed, the daily return data for hedge funds does not follow a normal distribution and has positive kurtosis. With this in mind, we are now sure that we have to take into account higher moments of the distribution when considering hedge fund returns. Our methodology, described in Section 2, comes in very handy at this point, as we are able to evaluate an aspect of the return distribution of the utmost importance: tail risk.

4.2 Results

4.2.1 Checkpoints

Using the methodology described in this paper, we obtain for each moving window described in Section 2: ten Lagrangian weights, one for each subcategory of hedge fund, representing the linear combination that produces the portfolio obtained in the dual optimization problem; thirty states for the Hellinger stochastic discount factor, one for each date in the moving window; and correspondingly, thirty states for the Hellinger risk-neutral probability.

Before presenting our tail risk series, we will present checkpoints that indicate if our empirical results are theoretically consistent. For that, we will pick the last moving window as an example for the reader, but keeping in mind that the results are fairly stable throughout the sample, and most importantly, that we checked for consistency considering all different windows.

First, we check that the stochastic discount factors are all positive and, there-

---

[7]We have used the Matlab code provided by Trujillo-Ortiz et al. (2007) to execute Royston's test for normality.

fore, that the no-arbitrage condition is being satisfied. Their positivity comes from the fact that they come from solutions to (strictly increasing) utility maximization problems. Below we present, as an example, the SDF implied using the last window of data.

Table 2
Hellinger's Implied SDF and RN Probabilities – Last Window of Data

Period	SDF	RPN	Period	SDF	RPN
1	1.2474	0.0416	16	0.9874	0.0329
2	1.2913	0.0431	17	1.7820	0.0594
3	1.3680	0.0456	18	0.2963	0.0099
4	0.5537	0.0185	19	1.2530	0.0418
5	0.2455	0.0082	20	1.8691	0.0623
6	0.2880	0.0096	21	1.4165	0.0472
7	0.4888	0.0163	22	1.8778	0.0626
8	0.3753	0.0125	23	0.8121	0.0271
9	0.9245	0.0308	24	0.2948	0.0098
10	2.3638	0.0788	25	0.5518	0.0184
11	0.7091	0.0236	26	0.5148	0.0172
12	0.2114	0.0071	27	0.6987	0.0233
13	0.1779	0.0059	28	0.7522	0.0251
14	3.3520	0.1118	29	1.4779	0.0493
15	1.6089	0.0537	30	0.1943	0.0065

We can also observe (see Figure 1) that the stochastic discount factor follows a hyperbolic structure. This structure is directly related to the proposed Cressie-Read family of discrepancy functions, as shown by Almeida and Garcia (2016), it restricts the analysis to a specific subset of admissible SDFs that captures non-linear patterns of returns via hyperbolic functions of returns.

For the risk neutral probabilities, we have that $\pi_i^{RN} = \frac{m_i^{MD}(1+r_f)}{T}$ and in all of the moving windows the risk-neutral probabilities sum to one. We present the result for the last moving window on Table 2. Some risk-neutral probabilities are higher, depending on the risk scenario, and others are lower, but we can guarantee in our model, by construction, that they sum to one in a given moving window.

Moreover, the same hyperbolic shape that we found for the SDFs can be observed in the structure of the risk-neutral probabilities (see Figure 2), since the last object is simply a normalization of the former. It is important to notice, at this point, that this hyperbolic structure is not valid for the returns on individual funds or types of funds. It only works for the endogenous portfolio, which can be obtained by multiplying the matrix of returns by the weights obtained from the Lagrangian and the result by $gamma = -\frac{1}{2}$. It is very clear from the graphic that physical and risk-neutral probabilities differ a lot, especially for tail returns. In addition, it is also interesting to note that lower returns receive more weight when we use these risk-neutral probabilities, while higher returns are attached to lower probabilities.
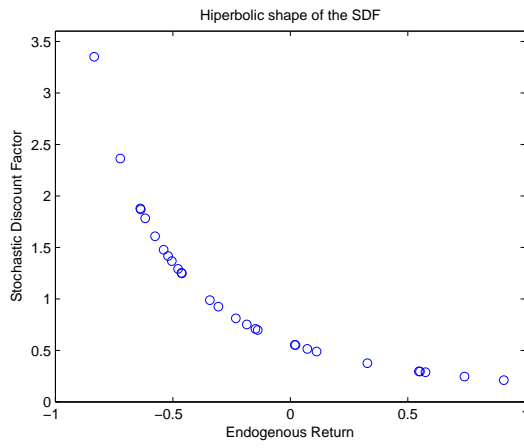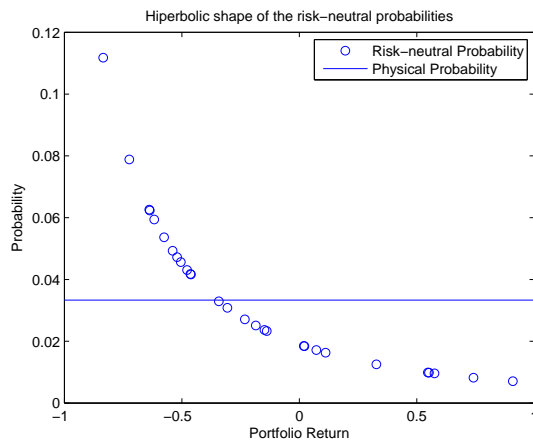
Figure 1
Hyperbolic Shape of Hellinger's SDF

Figure 2
Hyperbolic Shape of Hellinger's RN Probabilities

4.2.2 Tail risk time series

As interestingly put by Fernandes and Santos (2015), hedge fund managers can often pursue "picking up nickels in front of a steamroller" strategies. That is to say that they seek ways to produce high alphas by investing in tail risk. These strategies tend to produce small positive gains most of the time, but may also sporadically result in huge losses.

Throughout this paper, we have defended that it is of the utmost importance to take into account tail risk when analyzing hedge funds and we have presented a very straightforward methodology to do so. Below, we plot our tail risk time series for the aggregate hedge fund index, which includes all the ten subcategories of hedge funds.

The tail risk time series is very volatile and it might be difficult to distinguish whether tail risk is trending higher or lower just by looking at it. Therefore, we have used the Hodrick-Prescott filter to smooth the time series in order to reduce sensitivity and emphasize long term fluctuations instead of short term patterns. We have used the HP parameter as 14400, which is the regular value used for monthly analysis. The HP filter corresponds to the line shown in red in the figure. Our tail risk measure is the blue line.

Figure 3
Aggregate Hedge Fund Tail Risk
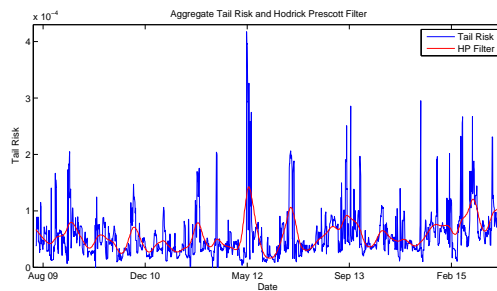
Furthermore, we would like to know how our hedge fund tail risk measure compares with total market volatility, and a tail risk measure for the Brazilian market index Ibovespa, which will be introduced in the next section.

First, we have to define the way by which we are measuring volatility. In this paper, we have used an Exponentially Weighted Moving Average (EWMA) model to obtain volatility for the market index IBOV. The initial value used in the EWMA model was 40.52%, corresponding to an approximate standard deviation for the first observations. The lambda used in the model was approximately 0.9143, which corresponds to the value that minimizes the daily mean squared errors between the squared returns for the IBOV Index and the EWMA estimate within

our sample. Below, we plot the volatility time series for the IBOV Index.[8]

Figure 4
An Estimate for the Volatility of the Ibovespa Index
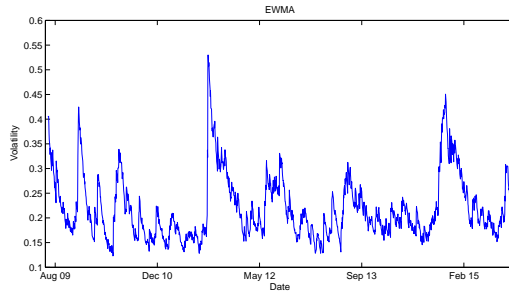
To see how our tail risk measure relates to total market volatility, we obtain the very down-to-earth measure of correlation. The correlation between the two time series is 0.1368.[9] As expected, this number is positive, indicating that the tail risk measure tends to move in the same direction as market volatility. However, it is rather low, telling us that the two measures contain different information. This is not difficult to reconcile since tail risk, coming from the prices of out-of-the money puts must contain some additional information about jumps in return, not directly captured by the EWMA measure of volatility.

If, on the other hand, we compare the HP filtered tail risk time series with the market volatility, we find a correlation of 0.7768, which is rather high. This tells us that the tendency for our tail risk time series moves along with the IBOV Index volatility. Even though it is not an obvious result, it is rather intuitive to have higher tail risk when volatility is higher. High volatility means that we get more spread out returns. In turn, this can lead to a higher chance of getting very low returns and, consequently, to increased tail risks.

In the next section, we offer measures of Tail risk implied for each individual hedge fund category and briefly analyze their correlation structure and also their correlations with Hellinger's implied Ibovespa tail risk.

4.2.3 Tail risk for individual subcategories of Hedge Funds (Within the Aggregate Framework)

The next three pictures present tail risk for the ten different hedge fund categories. Note that by just looking at those pictures we conclude that their behavior

---

[8]The original return time series for the IBOV Index can be found on Appendix B.

[9]Of course, since we are talking about tail events, that a linear measure like correlation should not be the most appropriate one to compare Tail risk to volatility. On the other hand, it gives an idea of how much they vary together on average, on a daily basis.

varies a lot across subcategories. Some categories like "Interest and Currency", "Fund of Funds", "Macro", and "Trading" have clearly a smallest number of spikes in the series than the others. Despite the fact that they do have a smaller number of spikes, those spikes usually achieve values two to three times higher than the average value of the corresponding tail risk time series, indicating a large variability of the tail risk measure. The tail risk series of the remaining categories have a large number of jumps and some of these jumps are very extreme. For instance, the Multi-strategy category presents jumps four to five times higher than the average tail risk of this category.

Figure 5
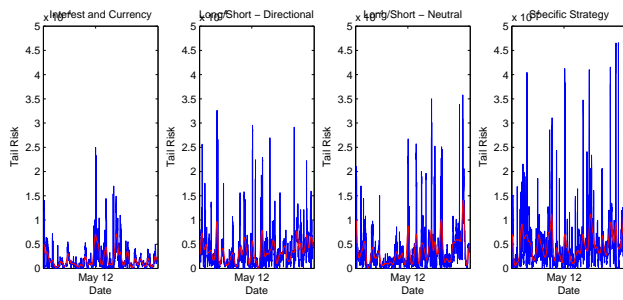Tail Risk for Individual Hedge Fund Categories – Part I

Figure 6
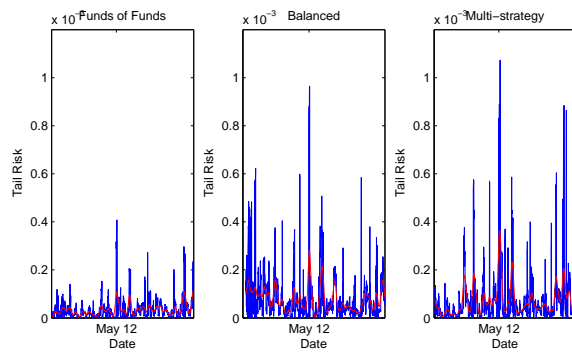Tail Risk for Individual Hedge Fund Categories – Part II

Table 3 presents the correlation structure of these individual series of tail risk as well as the corresponding correlations between these series and the aggregate measure of tail risk derived above. Since each of the individual measures is calculated by projecting, on each window of data, the implied SDF on the specific

Figure 7
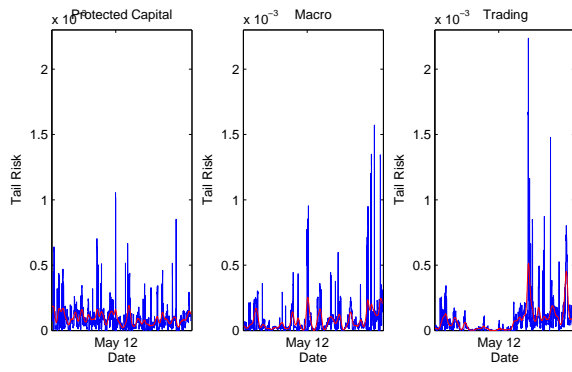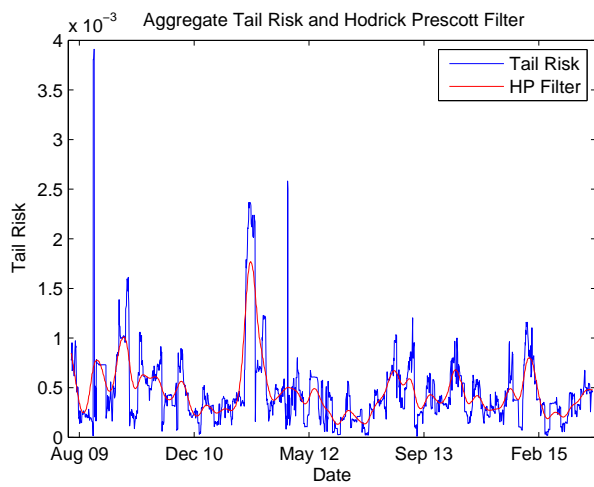Tail Risk for Individual Hedge Fund Categories – Part III

Figure 8
Ibovespa Hellinger's Tail Risk

sub-category being analyzed, the correlation matrix will indicate how aligned (or not) are the individual returns under the Hellinger risk neutral probability measure obtained with the implied Lagrange multipliers that solve the HARA utility problem described in the methodological section. We can see that, in general, most different subcategories have implied tail risk that is positively correlated across subcategories. An interesting point to mention is that categories like "Protected Capital", "LS-Neutral", and "Specific Strategy" that try to somehow work with hedging schemes indeed present the overall smallest correlations with respect to the tail risk of other subcategories. On the other hand, clearly "Balanced", "Macro", "Multi-strategy", and "Fund of Funds" appear to be among the categories most exposed to tail risk, both at individual level as well as at aggregate level.[10]

Indeed, since our aggregate measure of tail risk is obtained by an equally-weighted average of the ten subcategories, by looking at the last line of the previous table we are able to understand which subcategories contribute more to the aggregate tail risk. We quickly observe that the categories Balanced, Macro, Multi-Strategy and Fund of Funds, are the ones with the largest exposure to the aggregate tail risk index, with correlations close to 70% (except for the Fund of Funds, which presents a correlation of 61.91%). On the other hand, categories like LS-Directional and Interest and Currency have much smaller correlations, of the order of 30%. The interpretation here is easy: In terms of contribution to the aggregate measure (or index) of hedge fund tail risk, LS-Directional and Interest and Currency are the most idiosyncratic categories presenting less dependent extreme behavior than the remaining categories.

We also compare our measures of hedge fund tail risk, both aggregate as well as for each individual category, to the corresponding Hellinger's implied measure of tail risk for the Ibovespa.

By looking at the table above, we can see that the "Balanced", and "Protected Capital" strategies are the ones that present highest correlation with the Ibovespa tail risk, all others presenting correlations smaller than 20%, and in most cases very close to zero. The aggregate measure of hedge fund tail risk has a 9.03 % correlation with the Ibovespa tail risk. Since those measures of tail risk capture in part the behavior of jumps **expected** by investors within each of these different series of returns, we can see that expectations are very different across hedge fund returns and the Ibovespa aggregate market return index. In part, this should be expected since hedge funds have in general dynamic portfolios generating trading strategies whose dynamics is much richer than that of an aggregate equity index. Therefore, a nonlinear transformation of the returns implied by such strategies may potentially be very different from the corresponding transformation obtained for the aggregate equity index.

---

[10]Of course, our analysis is just of qualitative order since exposure to tail risk, with tail risk being clearly a nonlinear measure based on the left extreme of the returns' distribution, can not be properly measured by a linear relationship like correlation.

Table 3
Tail Risk Correlation table for individual hedge fund categories

	207	208	327	328	329	330	331	332	333	334	Total
207	1.0000	0.3383	0.2264	0.1696	0.3176	0.1147	0.4479	0.5557	0.2994	0.1362	0.6978
208	0.3383	1.0000	0.0973	0.0426	0.1309	-0.0144	0.1981	0.2737	0.1356	0.1056	0.4637
327	0.2264	0.0973	1.0000	0.1798	0.1468	0.2325	0.1178	0.2517	0.1180	-0.0060	0.3842
328	0.1696	0.0426	0.1798	1.0000	0.1480	0.0951	0.0839	0.1034	0.1150	0.0997	0.2960
329	0.3176	0.1309	0.1468	0.1480	1.0000	0.0944	0.4721	0.2897	0.1285	0.1718	0.6579
330	0.1147	-0.0144	0.2325	0.0951	0.0944	1.0000	0.0077	0.1624	-0.0184	0.0698	0.4588
331	0.4479	0.1981	0.1178	0.0839	0.4721	0.0077	1.0000	0.4302	0.2259	0.1799	0.6618
332	0.5557	0.2737	0.2517	0.1034	0.2897	0.1624	0.4302	1.0000	0.2390	0.1553	0.6191
333	0.2994	0.1356	0.1180	0.1150	0.1285	-0.0184	0.2259	0.2390	1.0000	-0.0158	0.2948
334	0.1362	0.1056	-0.0060	0.0997	0.1718	0.0698	0.1799	0.1553	-0.0158	1.0000	0.3439
Total	0.6978	0.4637	0.3842	0.2960	0.6579	0.4588	0.6618	0.6191	0.2948	0.3439	1.0000

Table 4

Relation between classification number and category

207 Balanced	208 Protected Capital	327 LS-Neutral	328 LS-Directional	329 Macro
330 Trading	331 Multi-Strategy	332 Funds of Funds	333 Interest and Currency	334 Specific Strategy

Table 5

Tail risk correlations between individual hedge fund categories and Ibovespa

207 Balanced	208 Protected Capital	327 LS-Neutral	328 LS-Directional	329 Macro
0.553	0.312	-0.044	0.044	0.105
330 Trading	331 Multi-Strategy	332 Funds of Funds	333 Interest and Currency	334 Specific Strategy
0.132	0.078	0.178	0.033	0.159

All in all, the most important contribution of the present paper was to introduce a tail risk measure for hedge funds in Brazil. We believe that it is important for both managers and financial regulators to have an objective measure of tail risk. The former needs to know whether the high returns are present due to compensation for the probability of extreme events, which the manager eagerly wishes to avoid. And the latter is interested in constraining global risk to avoid huge market downfalls that might affect the aggregate economy. We find that our measure is able to fill in the existing gap of measuring tail risk for these hedge funds in Brazil, especially considering the lack (and/or small number) of options data available for this market.

5. Conclusion

The results in this paper support the perception that hedge funds are structurally exposed to tail risk. We measure tail risk for Brazilian hedge funds building on the methodology proposed by Almeida and Garcia (2015). It relies on solving the dual problem of minimizing a discrepancy function. In our case, this discrepancy function was of the Cressie-Read type, with gamma $= -\frac{1}{2}$.

Solving this minimization problem, we find closed-form formulas for both Lagrangian weights[11] and stochastic discount factors for a sequence of moving windows. The SDFs are later used to compute risk neutral probabilities for each of the scenarios being considered. Then, based on the previously obtained RN probabilities, we build synthetic puts for each of the hedge fund subcategories to construct our tail risk measure.

Since Brazil's interest rate (one of our inputs) has changed a lot during the

---

[11]The Lagrange multipliers represent the weights given to each of the individual hedge fund categories when solving the dual maximization problem of the HARA utility function.

observed period, we have incorporated the risk-free interest rate time series into our analysis. This precludes the use of comparisons between different interest rate scenarios and solves the problem of having an interest rate sensitive tail risk.

The methodology that we have used to build our tail risk measure has one very important characteristic: It does not depend on options data. We use nothing but the daily returns on hedge funds. Therefore, it can be extended to calculate tail risk in different markets on which options data are not easily available. All in all, we have presented a very useful tool for both managers and regulators to control the exposure of portfolios and institutions to extreme events.

References

Adrian, T., Brunnermeier, M. K., & Nguyen, H.-L. Q. (2011). Hedge fund tail risk. In *Quantifying Systemic Risk*, pages 155–172. University of Chicago Press.

Agarwal, V. & Naik, N. Y. (2002). Characterizing systematic risk of hedge funds with buy-and-hold and option-based strategies. *London Business School.*

Almeida, C., Ardison, K., Garcia, R., & Vicente, J. (2016). Nonparametric tail risk stock returns and the macroeconomy. *Working paper, EDHEC Business School and EPGE/FGV.*

Almeida, C. & Garcia, R. (2013). Robust assessment of hedge fund performance through nonparametric discounting. *Working paper, EDHEC Business School and EPGE/FGV.*

Almeida, C. & Garcia, R. (2015). Economic implications of nonlinear pricing kernels. *Forthcoming at Management Science.*

Bailey, W., Li, H., & Zhang, X. (2004). Hedge fund performance evaluation: A stochastic discount factor approach. Technical report, Cornell University.

Bansal, R. & Lehman, B. (1997). Growth-optimal portfolio restrictions on asset pricing models. *Macroeconomic Dynamics*, 1:333–354.

Barro, R. (2006). Rare disasters and asset markets in the twentieth century. *The Quarterly Journal of Economics*, 121:823–866.

Bollerslev, T. & Todorov, V. (2011). Tails, fears, and risk premia. *The Journal of Finance*, 66(6):2165–2211.

Borwein, J. & Lewis, A. (1991). Duality relationships for entropy-like minimization problems. *SIAM Journal of Control and Optimization*, 29:325–338.

Capocci, D. & Hübner, G. (2004). Analysis of hedge fund performance. *Journal of Empirical Finance*, 11(1):55–89.

Carhart, M. (1997). On persistence in mutual fund performance. *Journal of Finance*, 52:57–82.

Cerny, A. (2003). Generalized sharpe ratios and asset pricing in incomplete markets. *European Finance Review*, 7:191–233.

Cressie, N. & Read, T. (1984). Multinomial goodness-of-fit tests. *Journal of the Royal Statistics Society Series B*, 46:440–464.

Duffie, D. (2001). *Dynamic Asset Pricing.* Princeton University Press, Princeton, NJ.

Fama, E. F. & French, K. R. (1998). Value versus growth: The international evidence. *Journal of Finance*, pages 1975–1999.

Fernandes, M. & Santos, E. A. M. (2015). Tail risk in the hedge fund industry. *Working Paper, QMUL and EESP-FGV*.

Ferson, W. (2003). Tests of mulitfactor pricing models, volatility bounds and portfolio performance. *Handbook of the Economics of Finance, Elsevier Science Publishers*, pages 743–800.

Hansen, L. & Jagannathan, R. (1991). Implications of security market data for models of dynamic economies. *Journal of Political Economy*, 99:225–262.

Hansen, L. & Jagannathan, R. (1997). Assessing specification errors in stochastic discount factor models. *Journal of Finance*, 52:557–590.

Kelly, B. & Jiang, H. (2012). Tail risk and hedge fund returns. *Chicago Booth Research Paper*, pages 12–44.

Kelly, B. & Jiang, H. (2014). Tail risk and asset prices. *Review of Financial Studies*, 27(10):2841–2871.

Kitamura, Y., Otsu, T., & Evdokimov, K. (2013). Robustness, infinitesimal neighborhoods, and moment restrictions. *Econometrica*, 81(3):1185–1201.

Kondor, P. (2009). Risk in dynamic arbitrage: The price effects of convergence trading. *Journal of Finance*, 64(2):631–655.

Mehra, R. & Prescott, E. (1985). The equity premium: A puzzle. *Journal of Monetary Economics*, 15(2):145–161.

Rietz, T. (1988). The equity premium: A solution. *Journal of Monetary Economics*, 22:117–131.

Siriwardane, I. (2013). The pricing of disaster risk. Technical report, NYU Stern School of Business.

Snow, K. (1991). Diagnosing asset pricing models using the distribution of asset returns. *Journal of Finance*, 46:955–983.

Stein, J. C. (2009). Presidential address: Sophisticated investors and market efficiency. *Journal of Finance*, 64:1517–1548.

Stutzer, M. (1995). A bayesian approach to diagnosis of asset pricing models. *Journal of Econometrics*, 68:367–397.

Trujillo-Ortiz, A., Hernandez-Walls, R., Barba-Rojo, K., & Cupul-Maganad, L. (2007). Roystest: Royston's multivariate normality test (matlab file).

Appendix A

According to ANBIMA, hedge funds can be rated as:

1. Balanced: funds that seeks long-term return by investing in a diversified portfolio (fixed income, stocks, exchange rate, etc.) with the possibility of rebalancing in the short-term.

2. Protected Capital: funds that seek return while protecting the principal invested, partially or totally. This type of fund does not admit leverage.

3. Specific Strategy: funds that adopt an investment strategy that implies specific risks, such as commodity risk or future index risk. Leverage is permitted.

4. Interest and Currency: funds that seek long-term return by investing on fixed income while admitting strategies that imply interest risk, price index risk and currency risk. Strategies that include exposure to variable income (stocks, etc.) are excluded. Leverage is permitted.

5. Long and Short – Directional: funds that operate assets and derivatives on the variable income market, using both long and short positions.

6. Long and Short – Neutral: funds that operate assets and derivatives, using both long and short positions, with the objective of keeping neutral exposure to the stock market.

7. Multi-strategy: funds that can adopt more than one investment strategy, without committing to a particular strategy. Leverage is permitted.

8. Macro: funds define strategies based on medium and long-term macroeconomic scenarios. These funds operate very diverse classes of assets and act in a directional manner. Leverage is permitted.

9. Funds of funds: have the objective of investing in more than one fund, managed by distinct people. The main competence of this type of fund is selecting other funds. Leverage is permitted.

10. Trading: funds that concentrate their investment strategies in different markets or asset classes, exploring opportunities originated by short-term movement on asset prices.

Appendix B

Below, we present the time series for daily returns for all the ten subcategories of hedge funds. We have separated them into two distinguished groups: those with lower range of return and those with higher range of return.

Amongst the funds with a narrow range of daily returns, we can find: Long and Short – Neutral, Long and Short – Directional, Funds of Funds and Interest and Currency.
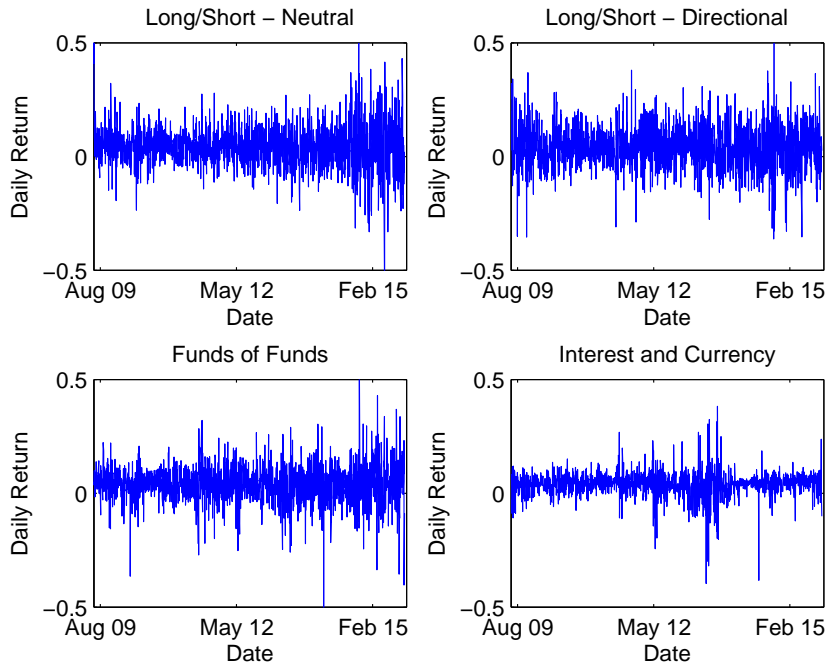
Likewise, amongst the funds with a wider range of daily returns, we can find: Balanced, Protected Capital, Macro, Trading, Multi-strategy and Specific Strategy.
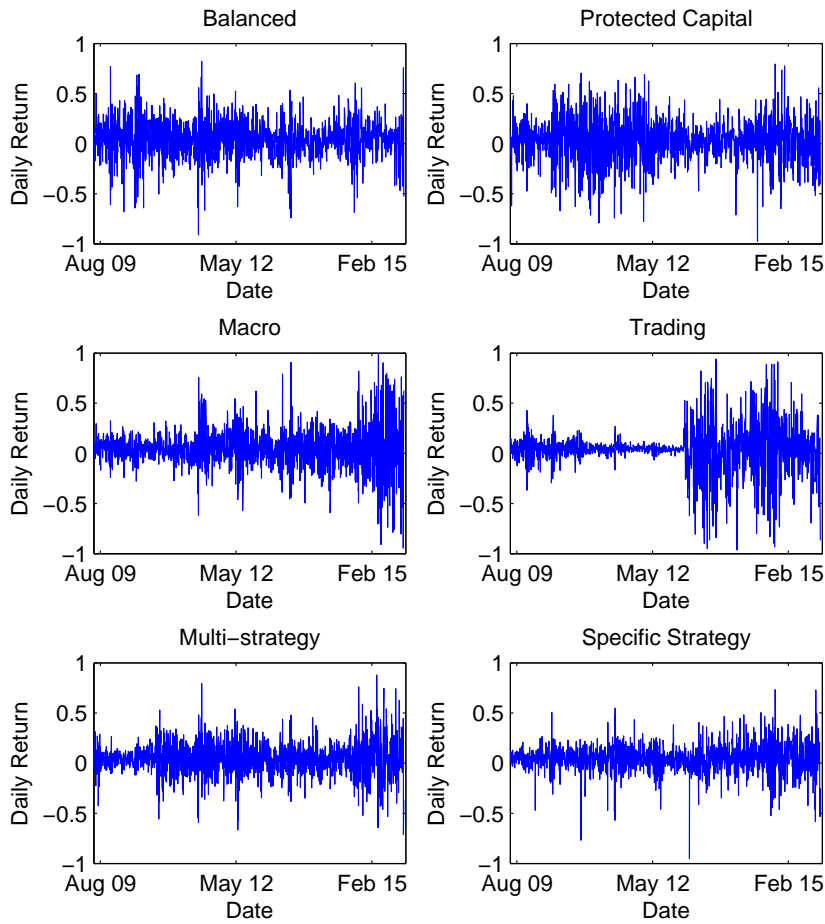
Last, but not least, we present the return time series for the IBOV Index.