A Simple Procedure for the Numerical Solution of Discrete Time Linear Two-Point Boundary-Value Problems with Finite Horizon and its use for Simulating Discrete Time Linear Models under Perfect Foresight*

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Summary: 1. Introduction; 2. General formulation of the discrete time linear two-point boundary-value problem with finite horizon; 3. A simple numerical solution procedure; 4. Special cases; 5. Two simple examples; 6. Concluding remarks.
Key words: simulation; two-point boundary-value problems; discrete time linear models; perfect foresight.
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This paper presents a simple procedure for the numerical solution of discrete time linear two-point boundary-value problems with finite horizon. The paper illustrates its use for simulating discrete time linear models under perfect foresight. For that purpose two simple models were chosen from the standard macroeconomic literature. The outstanding feature of the proposed procedure is its simplicity.

Este artigo apresenta um simples procedimento para a solução numérica de problemas lineares discretos de condições de contorno em dois pontos com horizonte finito. O artigo ilustra o seu uso para a simulação de modelos lineares discretos com previsão perfeita. Para este efeito dois modelos simples foram escolhidos da literatura macroeconômica padrão. O caráter excelente do algoritmo proposto consiste na sua simplicidade.

1. Introduction

In models with rational economic agents who form their expectations taking into account all the relevant information, the actual values of the endogenous variables not only depend on the past values of the variables of the model, but also on the expected future paths of the endogenous variables and these on the anticipated evolution of the exogenous variables of the model.

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In general, the models with perfect foresight contain – apart from the exogenous variables – both predetermined state variables whose current values are inherited from the past, and non-predetermined state variables, also called jumping variables, whose current values can be determined from the structure of the model and the available information on the present values and the future evolution of all the variables of the model.

Contrary to the simulation of a traditional dynamic model in which all the state variables are predetermined, the solution of a model under perfect foresight has to overcome the problem that there only exist initial values for the predetermined state variables. Moreover, different initial values for the non-predetermined state variables give rise to different dynamic adjustment paths, all of which are compatible with the postulate of perfect foresight. Nevertheless, in a well-specified linear model only one dynamic path will converge to the final long-run equilibrium, all other paths will diverge (Blanchard & Kahn, 1980).

Referring to a variant of Samuelson's "correspondence principle", it is often assumed that economic agents not only possess myopic perfect foresight, but also long-run perfect foresight, and that their rational behaviour makes them choose the convergent path, discarding the divergent paths. From a formal point of view, this assumption is equivalent to the introduction of a final or convergence condition for the non-predetermined state variables in substitution for the missing initial values.

This brief description suggests that the simulation of a dynamic perfect foresight model corresponds to solving a so-called two-point boundary-value problem. For this reason we present, in the following section, a general formulation of the discrete time linear two-point boundary-value problem with finite horizon. In section 3 we propose a simple procedure for the numerical solution of this type of problems, and in section 4 we consider several special cases. Although, presently, most macroeconomic models have infinite horizon, it is still possible to apply the procedures proposed in this essay. In an infinite horizon context, one has to choose a final period that is sufficiently far away, and the terminal values of the non-predetermined variables have to be taken as equal to their final long-run equilibrium or stationary values. In section 5 we briefly discuss the use of the proposed procedure for the simulation of discrete time linear models under perfect foresight. For that purpose we choose two simple illustrative models from the standard macroeconomic literature. The paper concludes with some final remarks.[1]

2. General Formulation of the Discrete Time Linear Two-Point Boundary-Value Problem with Finite Horizon

In what follows we shall only consider dynamic models that can be written in the form of the following linear first order matrix difference equation:

$$\omega_{t+1} = A\omega_t + Gz_t \tag{1}$$

where:

$\omega =$ column vector containing n endogenous state variables;

$z =$ column vector containing q exogenous variables;

$t =$ time;

the constant matrices A and G must have the dimensions $n \times n$ and $n \times q$, respectively.

The general form (1) may represent the dynamic part of the reduced form of a model written, e.g., in the structural form

$$F_1\omega_{t+1} + F_2\omega_t + F_3\nu_t + F_4 z_t = 0 \tag{2}$$

$$F_5\omega_{t+1} + F_6\omega_t + F_7\nu_t + F_8 z_t = 0 \tag{3}$$

where ν is the column vector containing m short-run endogenous variables. The constant matrices F_1 to F_8 must have the dimensions:

$$F_1 : n \times n, \quad F_2 : n \times n, \quad F_3 : n \times m, \quad F_4 : n \times q$$

$$F_5 : m \times n, \quad F_6 : m \times n, \quad F_7 : m \times m, \quad F_8 : m \times q$$

The structural form (2)–(3) can be seen as an adaptation of the standard structural form by Austin and Buiter (1982) and Buiter (1984) to the discrete time case.

[1] *The files containing our own simulation routines in the Matlab syntax, as well as a small documentation, are available on request.*

If the matrices F_7 and $F_1 - F_3 F_7^{-1} F_5$ are invertible, it is possible to obtain, by means of a few transformations, the reduced form

$$\omega_{t+1} = A\omega_t + Gz_t \tag{4}$$

$$\nu_t = R\omega_t + Sz_t \tag{5}$$

where:

$$A = -\left(F_1 - F_3 F_7^{-1} F_5\right)^{-1} \left(F_2 - F_3 F_7^{-1} F_6\right) \tag{6}$$

$$G = -\left(F_1 - F_3 F_7^{-1} F_5\right)^{-1} \left(F_4 - F_3 F_7^{-1} F_8\right) \tag{7}$$

$$R = -F_7^{-1} \left(F_6 + F_5 A\right) \tag{8}$$

$$S = -F_7^{-1} \left(F_8 + F_5 G\right) \tag{9}$$

As the values of the short-run endogenous variables, contained in ν, can easily be computed by means of (5), we shall concentrate in what follows on the solution of the first order linear matrix difference equation (4).

Note also that considering only first order matrix difference equations does not constitute a limitation of the generality. E.g., the second order matrix difference equation

$$\omega_{t+1} = A_1\omega_t + A_2\omega_{t-1} + Gz_t \tag{10}$$

can be written, with the help of the auxiliary variables

$$\omega L_{t+1} = \omega_t, \tag{11}$$

as the first order matrix difference equation

$$\begin{pmatrix} \omega_{t+1} \\ \omega L_{t+1} \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ I & 0_1 \end{pmatrix} \begin{pmatrix} \omega_t \\ \omega L_t \end{pmatrix} + \begin{pmatrix} G \\ 0_2 \end{pmatrix} z_t \tag{12}$$

where I = identity matrix and 0_1, 0_2 = zero matrices of adequate dimensions.

Consider first the case in which the vector ω of the standard form (1) contains only predetermined state variables, included in the vector x. If we know the initial values of x in $t = 0$, $x_0 = \overline{x}_0$, we have the traditional initial value problem:

$$x_{t+1} = Ax_t + Gz_t, \qquad x_0 = \overline{x}_0 \tag{13}$$

Given the values of z_t, $t = 0, 1, \cdots, T-1$, as well as the matrices A and G, the values of x_t, $t = 1, 2, \cdots, T$, can easily be computed recursively.

Consider now the case in which the vector ω of the standard form (1) contains n_1 predetermined state variables and $n_2 = n - n_1$ non-predetermined state variables (or jumping variables). Without loss of generality, suppose that the state variables be ordered as indicated, so that we can write the vector ω as

$$\omega = \begin{pmatrix} x \\ y \end{pmatrix} \tag{14}$$

where x = column vector containing n_1 predetermined state variables and y = column vector containing n_2 non-predetermined state variables. If we know the initial values of the predetermined state variables, $x_0 = \overline{x}_0$, as well as the terminal values of the non-predetermined state variables, $y_T = \overline{y}_T$, we have the two-point boundary-value problem:

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = A \begin{pmatrix} x_t \\ y_t \end{pmatrix} + Gz_t, \qquad x_0 = \overline{x}_0, \qquad y_T = \overline{y}_T \tag{15}$$

In this case it is obviously not possible to apply immediately the recursive solution procedure as in the case of (13), as we do not (yet) know the initial values of y in $t = 0$, y_0. Note that in order to emulate the simulation of a model with infinite horizon, one has to choose a final period, T, that is sufficiently far away, using as terminal values of the non-predetermined state variables, $y_T = \overline{y}_T$, the values of those variables in the final long-run equilibrium.

A closer look at the formulation (15) reveals that it is implicitly assumed that we know the terminal values of all the non-predetermined state variables and that no predetermined state variable is subject to a final value restriction. It may, nevertheless, occur that some predetermined state variables must fulfill not only an initial value restriction, but also a terminal value restriction, whereas there may be non-predetermined state variables that are not even subject to a terminal condition. This suggests the possibility of generalizing the formulation (15), considering the following four types of state variables:

a) n_1' predetermined state variables, subject only to an initial value restriction – they are contained in the column vector $x1$, whose initial value in $t = 0$ is $x1_0 = \overline{x1}_0$;

b) n_1'' predetermined state variables, subject to an initial value and a terminal value restriction – they are contained in the column vector $x2$, whose initial value in $t = 0$ is $x2_0 = \overline{x2}_0$ and whose terminal value in $t = T$ is $x2_T = \overline{x2}_T$;

c) n_2' non-predetermined state variables, subject only to a terminal value restriction – they are contained in the column vector $y1$, whose terminal value in $t = T$ is $y1_T = \overline{y1}_T$;

d) n_2'' non-predetermined state variables, subject neither to an initial nor to a terminal value restriction – they are contained in the column vector $y2$.

Introducing the four categories of state variables in the indicated order, we can write the vector ω as

$$\omega = \begin{pmatrix} x1 \\ x2 \\ y1 \\ y2 \end{pmatrix} \tag{16}$$

obtaining the following formulation of the general discrete time linear two-point boundary-value problem with finite horizon:

$$\begin{pmatrix} x1_{t+1} \\ x2_{t+1} \\ y1_{t+1} \\ y2_{t+1} \end{pmatrix} = A \begin{pmatrix} x1_t \\ x2_t \\ y1_t \\ y2_t \end{pmatrix} + Gz_t, \tag{17}$$

$$x1_0 = \overline{x1}_0, \;\; x2_0 = \overline{x2}_0, \;\; x2_T = \overline{x2}_T, \;\; y1_T = \overline{y1}_T$$

As (17) constitutes a system of n linear first-order difference equations, we need exactly n boundary conditions:

$$n_1' + 2n_1'' + n_2' = n \tag{18}$$

But, as $n_1' + n_1'' + n_2' + n_2'' = n$ by definition, we find that

$$n_1'' = n_2'' \tag{19}$$

Therefore, the number of predetermined state variables that are subjected to an initial as well as a final value restriction must be equal to the number of non-predetermined state variables that are not subjected to any boundary restriction.

3. A Simple Numerical Solution Procedure

In this section we propose a simple procedure for the numerical solution of problem (17), which will be presented in two versions. While the first one may be more easily understood, the second can more easily be generalized.

As regards the general idea, the proposed procedure consists of the following two steps:

a) first, the "missing" initial values of the non-predetermined state variables are computed, so that the two-point boundary-value problem can be transformed into a "traditional" initial value problem;

b) second, the initial value problem is solved recursively, using the corresponding matrix difference equation, starting from the initial values of all the state variables of the model.

3.1 First version of the procedure

From equation (17) we obtain by means of successive substitutions:

$$\omega_T = \alpha\omega_0 + \gamma \tag{20}$$

with

$$\alpha = A^T \tag{21}$$

$$\gamma = A^{T-1}Gz_0 + A^{T-2}Gz_1 + \cdots + A^2Gz_{T-3} + AGz_{T-2} + Gz_{T-1} \tag{22}$$

The vector ω is defined in (16).

In order to compute the "missing" initial values, we start from the equations for $x2_T$ and $y1_T$. Partitioning α and γ in conformity with ω and using the border conditions of (17), we get:

$$\begin{pmatrix} \overline{x2}_T \\ \overline{y1}_T \end{pmatrix} = \overline{\alpha}_1 \begin{pmatrix} \overline{x1}_0 \\ \overline{x2}_0 \end{pmatrix} + \overline{\alpha}_2 \begin{pmatrix} y1_0 \\ y2_0 \end{pmatrix} + \overline{\gamma} \tag{23}$$

with

$$\overline{\alpha}_1 = \begin{pmatrix} \alpha_{21} & \alpha_{22} \\ \alpha_{31} & \alpha_{32} \end{pmatrix}, \qquad \overline{\alpha}_2 = \begin{pmatrix} \alpha_{23} & \alpha_{24} \\ \alpha_{33} & \alpha_{34} \end{pmatrix}, \qquad \overline{\gamma} = \begin{pmatrix} \gamma_2 \\ \gamma_3 \end{pmatrix} \tag{24}$$

If $\overline{\alpha}_2$ is invertible, we finally obtain the following expression for the initial values of the non-predetermined state variables, $y1_0$ and $y2_0$:

$$\begin{pmatrix} y1_0 \\ y2_0 \end{pmatrix} = \overline{\alpha}_2^{-1} \left\{ \begin{pmatrix} \overline{x2}_T \\ \overline{y1}_T \end{pmatrix} - \overline{\alpha}_1 \begin{pmatrix} \overline{x1}_0 \\ \overline{x2}_0 \end{pmatrix} - \overline{\gamma} \right\} \tag{25}$$

Once we know the values for $y1_0$ and $y2_0$, we can easily compute the values of the endogenous variables $x1_t$, $x2_t$, $y1_t$, and $y2_t$, $t = 1, 2, \cdots, T$, recursively, starting from their initial values at $t = 0$ and making use of the matrix difference equation in (17).

3.2 Second version of the procedure

The second version of the procedure is based on Mattheij and Staarink (1992) and can be more easily generalized, as we will show in the following section.

This version starts from the same equation (20) and adds to it the matrix equation containing the border conditions,

$$M_1\omega_0 + M_2\omega_T = M_0 \tag{26}$$

with

$$M_1 = \begin{pmatrix} I_{n_1'} & 0 & 0 & 0 \\ 0 & I_{n_1''} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_2'} & 0 \\ 0 & I_{n_1''} & 0 & 0 \end{pmatrix},$$
$$M_0 = \begin{pmatrix} \overline{x1}_0 \\ \overline{x2}_0 \\ \overline{y1}_T \\ \overline{x2}_T \end{pmatrix} \tag{27}$$

and where I_{n_j} = identity matrix of dimension $n_j \times n_j$.

The expressions (20) and (26) form a linear equation system in the variables contained in ω_0 and ω_T which can be written as

$$\overline{M} \begin{pmatrix} \omega_0 \\ \omega_T \end{pmatrix} = \overline{L} \tag{28}$$

with

$$\overline{M} = \begin{pmatrix} -\alpha & I_n \\ M_1 & M_2 \end{pmatrix}, \qquad \overline{L} = \begin{pmatrix} \gamma \\ M_0 \end{pmatrix} \tag{29}$$

and where I_n = identity matrix of dimension $n \times n$. If $\overline{M}$ has full rank, we get:

$$\begin{pmatrix} \omega_0 \\ \omega_T \end{pmatrix} = \overline{M}^{-1}\overline{L} \tag{30}$$

Once we know ω_0, we can easily compute the values of ω_t, $t = 1, 2, \cdots, T$, recursively.

3.3 Generalization

As Mattheij and Staarink (1992:5) observe, the procedure of the last two sections, based on the principle usually characterized as "single shooting", generally gives inaccurate numerical results when – in the case under consideration – A^T contains elements with "very high" (absolute) values. In order to overcome this problem, it has been suggested to divide the simulation interval into subintervals. The principle behind these generalizations is usually characterized as "multiple shooting".

For the following exposition we assume that the simulation interval $[0, t_N]$ is divided into N subintervals $[0, t_1)$, $[t_1, t_2), \cdots, [t_{N-1}, t_N]$. Applying to each subinterval a solution of the type of (20), we find:

$$\begin{aligned} \omega_{t_1} &= \alpha^{(1)}\omega_0 + \gamma^{(1)} \\ \omega_{t_2} &= \alpha^{(2)}\omega_{t_1} + \gamma^{(2)} \\ \cdots \\ \omega_{t_N} &= \alpha^{(N)}\omega_{t_{N-1}} + \gamma^{(N)} \end{aligned} \tag{31}$$

The border conditions, rewritten as

$$M_1\omega_0 + M_2\omega_{t_N} = M_0 \tag{32}$$

and the expressions in (31) form a linear equation system

$$\overline{M} \cdot \begin{pmatrix} \omega_0 \\ \omega_{t_1} \\ \cdots \\ \omega_{t_N} \end{pmatrix} = \overline{L} \tag{33}$$

where

$$\overline{M} = \begin{pmatrix} -\alpha^{(1)} & I & 0 & \cdots & 0 & 0 \\ 0 & -\alpha^{(2)} & I & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -\alpha^{(N)} & I \\ M_1 & 0 & 0 & \cdots & 0 & M_2 \end{pmatrix}, \quad \overline{L} = \begin{pmatrix} \gamma^{(1)} \\ \gamma^{(2)} \\ \gamma^{(3)} \\ \cdots \\ \gamma^{(N)} \\ M_0 \end{pmatrix} \tag{34}$$

If $\overline{M}$ is of full rank, we can compute the values of ω_0, $\omega_{t_1}, \cdots, \omega_{t_{N-1}}$. Introducing these values into the dynamic equations corresponding to each subinterval, we can easily obtain the values of ω_t, $t = 1, 2, \cdots, T$.

For an adequate use of this generalized solution procedure, it is important to choose a convenient number of subintervals. Here we have an optimization problem. If we choose a relatively low value for N, the equation system corresponding to (33) will be of limited dimension, but the matrices $\alpha^{(i)}$ will have elements with relatively high numerical values, which will have a negative effect on the precision of the calculations. On the other hand, if we choose a relatively high value for N, we obtain an equation system of the type of (33) of high dimension, but the matrices $\alpha^{(i)}$ will have elements with relatively low numerical values, which enhances the precision of the calculations.

4. Special Cases

In this section we mention several special cases, some of which can be found fairly often in the literature.

Setting $n1'' = n2'' = 0$, we obtain the special case that corresponds to (15), which can be rewritten in the more general notation as:

$$\begin{pmatrix} x1_{t+1} \\ y1_{t+1} \end{pmatrix} = A \begin{pmatrix} x1_t \\ y1_t \end{pmatrix} + Gz_t, \qquad x1_0 = \overline{x1}_0, \qquad y1_T = \overline{y1}_T \tag{35}$$

In this case we have

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{13} \\ \alpha_{31} & \alpha_{33} \end{pmatrix} \Rightarrow \overline{\alpha}_1 = \alpha_{31}, \qquad \overline{\alpha}_2 = \alpha_{33} \tag{36}$$

and

$$\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_3 \end{pmatrix} \Rightarrow \overline{\gamma} = \gamma_3 \tag{37}$$

Many perfect foresight models can be written in this form, which, moreover, corresponds to the form used by Blanchard and Kahn (1980), as well as by McKibbin (1987) and McKibbin and Sachs (1991). The first illustrative example of the next section belongs to this category.

Setting $n1' = n2' = 0$, we have a second special case of the general problem:

$$\begin{pmatrix} x2_{t+1} \\ y2_{t+1} \end{pmatrix} = A \begin{pmatrix} x2_t \\ y2_t \end{pmatrix} + Gz_t, \qquad x2_0 = \overline{x2}_0, \qquad x2_T = \overline{x2}_T \tag{38}$$

with

$$\alpha = \begin{pmatrix} \alpha_{22} & \alpha_{24} \\ \alpha_{42} & \alpha_{44} \end{pmatrix} \Rightarrow \overline{\alpha}_1 = \alpha_{22}, \qquad \overline{\alpha}_2 = \alpha_{24} \tag{39}$$

and

$$\gamma = \begin{pmatrix} \gamma_2 \\ \gamma_4 \end{pmatrix} \Rightarrow \overline{\gamma} = \gamma_2 \qquad . \tag{40}$$

The second illustrative model of the next section belongs to this category.

Setting $n2' = 0$, we have the special case:

$$\begin{pmatrix} x1_{t+1} \\ x2_{t+1} \\ y2_{t+1} \end{pmatrix} = A \begin{pmatrix} x1_t \\ x2_t \\ y2_t \end{pmatrix} + Gz_t, \quad x1_0 = \overline{x1}_0, \quad x2_0 = \overline{x2}_0, \quad x2_T = \overline{x2}_T \tag{41}$$

with

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{24} \\ \alpha_{41} & \alpha_{42} & \alpha_{44} \end{pmatrix} \Rightarrow \overline{\alpha}_1 = (\alpha_{21}\ \alpha_{22}), \quad \overline{\alpha}_2 = \alpha_{24} \tag{42}$$

and

$$\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_4 \end{pmatrix} \Rightarrow \overline{\gamma} = \gamma_2 \tag{43}$$

Finally, setting $n1' = 0$, we have the special case:

$$\begin{pmatrix} x2_{t+1} \\ y1_{t+1} \\ y2_{t+1} \end{pmatrix} = A \begin{pmatrix} x2_t \\ y1_t \\ y2_t \end{pmatrix} + Gz_t,\ x2_0 = \overline{x2}_0,\ x2_T = \overline{x2}_T,\ y1_T = \overline{y1}_T \tag{44}$$

with

$$\alpha = \begin{pmatrix} \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{42} & \alpha_{43} & \alpha_{44} \end{pmatrix} \Rightarrow \overline{\alpha}_1 = \begin{pmatrix} \alpha_{22} \\ \alpha_{32} \end{pmatrix}, \quad \overline{\alpha}_2 = \begin{pmatrix} \alpha_{23} & \alpha_{24} \\ \alpha_{33} & \alpha_{34} \end{pmatrix} \tag{45}$$

and

$$\gamma = \begin{pmatrix} \gamma_1 \\ \gamma_3 \\ \gamma_4 \end{pmatrix} \Rightarrow \overline{\gamma} = \gamma_3 \tag{46}$$

5. Two Simple Examples

In this section we illustrate the use of the presented procedures for simulating discrete time linear models under perfect foresight. For that purpose we choose two simple models from the standard macroeconomic literature. Starting from the basic equations of each model, we show how to obtain the standard structural form and we characterize the variables of the model on basis of the model's assumptions.

5.1 The exchange-rate overshooting model by Dornbusch

The first example we shall consider is the well-known exchange-rate overshooting model by Dornbusch (1976). We shall discuss two versions of the same model, which only differ in the temporal specification of the price dynamics.

First version

Consider first the following simplified version of the discrete time version of the model, contained in Azariadis (1993:46-51):

$$m_t - p_t = \kappa - \lambda i_t \tag{47}$$

$$i_t = i^* + e_{t+1} - e_t \tag{48}$$

$$p_{t+1} - p_t = \alpha(d_t - \overline{q}) \tag{49}$$

$$d_t = \delta_0 + \delta_1(e_t - p_t) \tag{50}$$

where:

d = aggregate demand;

e = nominal exchange rate (defined as units of domestic currency per unit of foreign currency);

i = domestic nominal interest rate;

i^* = international nominal interest rate (assumed constant);

m = nominal money stock;

p = domestic price level;

t = time;

$\overline{q}$ = potential or full-employment output.

All the variables, except interest rates, are in logs. The coefficients are written in Greek letters, and all the slope coefficients are defined as positive. The assumption of perfect foresight with respect to the exchange rate is already contained in equation (48) in which it is supposed that $e^e_{t+1} = e_{t+1}$, with e^e_{t+1} = expected exchange rate for $t+1$.

The model (47)–(50) can be directly written in the standard structural form (2)–(3):

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{t+1} \\ e_{t+1} \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} p_t \\ e_t \end{pmatrix} + \begin{pmatrix} -\alpha & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} d_t \\ i_t \end{pmatrix} + \begin{pmatrix} \overline{\alpha}_q & 0 \\ i^* & 0 \end{pmatrix} \begin{pmatrix} 1 \\ m_t \end{pmatrix} = 0 \tag{51}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p_{t+1} \\ e_{t+1} \end{pmatrix} + \begin{pmatrix} \delta_1 & -\delta_1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} p_t \\ e_t \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} d_t \\ i_t \end{pmatrix} + \begin{pmatrix} -\delta_0 & 0 \\ -\kappa & 1 \end{pmatrix} \begin{pmatrix} 1 \\ m_t \end{pmatrix} = 0 \tag{52}$$

According to the assumptions of the model, p is the predetermined state variable and e is the non-predetermined state variable, d and i are the short-run endogenous variables and m is the exogenous variable. The constant terms are introduced as multiples of one. Therefore, we have: $x1_t = p_t$, $y1_t = e_t$, $\nu_t = [d_t\ i_t]'$ and $z_t = [1\ m_t]'$.

Second version

The second version of the model contains a slightly different temporal specification of the equation for the price dynamics:

$$p_t - p_{t-1} = \alpha(d_t - \overline{q}) \tag{53}$$

In order to write this version of the model in the standard structural form (2)–(3), we define the new price variable

$$p1_{t+1} = p_t \tag{54}$$

following Blanchard (1980), and rewrite the equation (53) as

$$p1_{t+1} - p1_t = \alpha(d_t - \overline{q}) \tag{55}$$

Using the equations (55), (48), (54), (50) and (47) in the indicated order, we finally obtain the standard structural form:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p1_{t+1} \\ e_{t+1} \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} p1_t \\ e_t \end{pmatrix} + \begin{pmatrix} 0 & -\alpha & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} p_t \\ d_t \\ i_t \end{pmatrix} + \begin{pmatrix} -\overline{\alpha}_q & 0 \\ i^* & 0 \end{pmatrix} \begin{pmatrix} 1 \\ m_t \end{pmatrix} = 0 \tag{56}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p1_{t+1} \\ e_{t+1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\delta_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p1_t \\ e_t \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ \delta_1 & 1 & 0 \\ -1 & 0 & \lambda \end{pmatrix} \begin{pmatrix} p_t \\ d_t \\ i_t \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\delta_0 & 0 \\ -\kappa & 1 \end{pmatrix} \begin{pmatrix} 1 \\ m_t \end{pmatrix} = 0 \tag{57}$$

being $x1_t = p1_t$, $y1_t = e_t$, $\nu_t = [p_t \; d_t \; i_t]'$ and $z_t = [1 \; m_t]'$.

Once the numerical values for the coefficients of the matrices of the standard structural form, the initial value of the predetermined state variable, the terminal value of the non-predetermined state variable and the path of the exogenous variable are chosen, it is possible to simulate the model in any of the two versions. Note that, in order to emulate the simulation of the model in an infinite horizon context, one has to choose a final period T that is sufficiently far away, using as terminal value of the non-predetermined state variable the value of that variable in the final long-run equilibrium.

5.2 A model for the trade balance and the current account by Sachs and Larrain

The second illustrative example is a simple model, inspired by Sachs and Larrain (1992, ch. 6), for the analysis of the trade balance and current account dynamics of a small open economy, subjected to different types of real (output) shocks. Contrary to the first example, this model possesses an explicit dynamic microfoundation. Dornbusch (1983) has a more elaborate model of the same type in which the small open economy under consideration produces and consumes both traded and non-traded goods.

It is supposed that the representative family maximizes the intertemporal utility function

$$-\frac{1}{2}\sum_{t=0}^{T-1}\beta^t(c_t-\bar{c})^2, \quad 0<\beta<1, \quad 0<c_t<\bar{c}, \qquad \forall t \tag{58}$$

subject to the budget constraint

$$b_{t+1}=b_t(1+r)+q_t-c_t \tag{59}$$

and the boundary conditions

$$b_0=0, \qquad b_T\geq 0 \tag{60}$$

where:

$b =$ net foreign assets (represented by the net tenancy of an internationally tradable short-run bond);

$c =$ consumption;

$r =$ (constant) foreign real interest rate;

$t =$ time;

q = domestic output (taken as exogenous);

β = discount factor.

The time horizon (= life of the representative family) is taken to be finite. The trade account surplus corresponds to the difference between output and consumption,

$$TB_t = q_t - c_t \tag{61}$$

whereas the current account surplus equals the surplus of the trade account plus the surplus of the interest account,

$$CA_t = TB_t + rb_t \tag{62}$$

where TB = trade account surplus and CA = current account surplus.

As discussed in Stokey and Lucas (1989:11), this model has a unique solution which is fully characterized by the Kuhn-Tucker conditions. To obtain these conditions, form the Lagrangean

$$\mathcal{L} = \sum_{t=0}^{T-1} \beta^t \left\{ -\frac{1}{2}(c_t - \bar{c})^2 + \mu_t \left[b_t(1+r) + q_t - c_t - b_{t+1} \right] \right\} \tag{63}$$

where μ = Lagrange multiplier. From the first order partial derivatives of (63) we obtain the first-order conditions:

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 \Rightarrow \bar{c} - c_t = \mu_t, \qquad t = 0, 1, \cdots, T-1 \tag{64}$$

$$\frac{\partial \mathcal{L}}{\partial b_{t+1}} = 0 \Rightarrow \mu_{t+1}\beta(1+r) = \mu_t, \qquad t = 0, 1, \cdots, T-2 \tag{65}$$

$$\frac{\partial \mathcal{L}}{\partial \mu_t} = 0 \Rightarrow b_{t+1} = (1+r)\,b_t + c_t - q_t, \qquad t = 0, 1, \cdots, T-1 \tag{66}$$

Moreover, taking into account that the objective function is strictly increasing (within the relevant interval), we have the boundary condition:

$$b_T = 0 \tag{67}$$

The equation system composed of the expressions (66), (65), (64), (61), and (62) can easily be written in the standard structural form (2)–(3):

$$\begin{pmatrix} 1 & 0 \\ 0 & \beta(1+r) \end{pmatrix} \begin{pmatrix} b_{t+1} \\ \mu_{t+1} \end{pmatrix} + \begin{pmatrix} -(1+r) & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} b_t \\ \mu_t \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_t \\ TB_t \\ CA_t \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ q_t \end{pmatrix} = 0 \tag{68}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b_{t+1} \\ \mu_{t+1} \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \\ -r & 0 \end{pmatrix} \begin{pmatrix} b_t \\ \mu_t \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} c_t \\ TB_t \\ CA_t \end{pmatrix} + \begin{pmatrix} -\overline{c} & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ q_t \end{pmatrix} = 0 \tag{69}$$

According to the assumptions of the model, b is the predetermined state variable with the boundary values $b_0 = b_T = 0$, and μ is the non-predetermined state variable. Finally, c_t, TB_t, and CA_t are the short-run endogenous variables. Therefore, we have: $x2_t = b_t$, $y2_t = \mu_t$, $\nu_t = [c_t\ TB_t\ CA_t]'$, and $z_t = [1\ q_t]'$.

We conclude that the simulation of the model, composed of the equations (58)–(62), can be considered as a special case of the two-point boundary-value problem (17).

6. Concluding Remarks

In this paper we presented a simple procedure for the numerical solution of discrete time linear two-point boundary-value problems with finite horizon, and we illustrated its use for the simulation of discrete time linear models under perfect foresight. For that purpose we selected two simple models from the standard macroeconomic literature. The attractive feature of the proposed procedure is its simplicity.

In its simple version, the proposed procedure is based on the "single shooting" principle and computes, in the first instance, the "missing" initial values of the non-predetermined state variables, so that the two-point boundary-value problem can be transformed into a "traditional" initial value problem. In its generalized version, the procedure is based on the "multiple shooting" principle. In this case, we compute, in the first instance, the initial values of the state variables at the beginning of each subinterval, so that the two-point boundary value problem can be transformed into a sequence of initial value problems.

The fact that the procedure is constructed for linear models constitutes a less important restriction than is generally thought. Many models can be linearized around a conveniently chosen point and often the simulation results do not seem to differ too much – at least qualitatively – from the results that can be generated (by means of other procedures) using the original non-linear version of the same model.

On the other hand, to obtain satisfactory numerical results, it is important to choose the adequate number of subintervals. In the case of simple models that are to be simulated in an interval that is not too large, the simple version of the procedure will often suffice. On the other hand, from a certain extension of the model or of the simulation interval on it, is recommended to use the generalized procedure, based on "multiple shooting". In this case the choice of an adequate number of subintervals is particularly important.

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