Applications of Real Options in the Real Estate Market Focusing the City of Rio de Janeiro*

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Abstract

An investment opportunity in the real estate market can be compared with an option and, because of this analogy, we can make use of the real options theory to determine not only the best timing to begin a new property development, but also to determine the optimal density to develop in a specific property. To investigate these two main decisions, we are going to base our research on the model developed by Williams (1991) and extend it in a way to include taxes and a discount in the net cash inflow caused by the time spent in the construction process. An empirical analysis of the case of the residential real estate market in the city of Rio de Janeiro is going to be developed in a way to verify the compatibility of the theoretical model developed here according to the reality of this market. We found that the extensions proposed to the basic model of Williams (1991) had significant effects in the theoretical model. We could also confirm empirically that the theoretical results are compatible to the reality of this market.

Key Words: Real options, real estate market, timing, density, taxes .

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Resumo

A oportunidade de investimento no mercado imobiliário pode ser comparada a uma opção e, devido a esta analogia, podemos utilizar a teoria de opções reais para determinar o momento ótimo para o início da construção de uma propriedade e a densidade ótima de construção. Para investigar estas duas principais decisões utilizaremos como modelo básico o trabalho de Williams (1991) estendido de forma a incluir impostos e um desconto no fluxo de caixa devido ao tempo gasto na construção. Uma análise empírica para o caso do mercado imobiliário residencial para cidade do Rio de Janeiro é realizada de forma a verificar a compatibilidade do modelo desenvolvido com a realidade. Verificamos que as extensões propostas ao modelo básico de Williams (1991) possuem impactos significativos no modelo. Nossa análise empírica confirma ainda que os resultados teóricos obtidos são compatíveis com a realidade do mercado em questão.

1. Introduction.

The investments in the real estate market are really important not only in Brazil, but also in many countries around the world. These investments are characterized by a great probability of creating high profits and, because of the absence of a formal tool as the one we are going to develop here, they are usually under evaluated. There are a lot of decisions that must be made in order for these investments to be optimally developed. The choice of the optimal time of the construction is one of the most important decisions. Uncertainties about the future costs of construction and market conditions, after the project is completed, are important features of this decision. This situation enables investors to defer the development and benefit from the resolution of uncertainty about market conditions and construction costs during this period. This opportunity is known in the literature as an option to defer investment and it is particularly valuable in real estate investments. The optimal density of construction or, in other words, the best number of units per area to develop in a specific property, is another important feature for the creation of an optimal investment.

The objective of this paper is to investigate the decisions of choosing the best timing to begin a new property development and the optimal density at which to develop this specific property making use of real options theory. We are going to base our research on the model developed by Williams (1991). This is a continuous time stochastic model where the owner of a vacant land, subject to legal limitations, has to decide the best timing and the optimal density to develop a new property. However, this model has the limitation of considering the construction process instantaneous. In practice, we know that an investor will use some time in the construction process and it is important to consider this time in the model. So, the first contribution of this paper is to extend this basic model in order to take into account "time to build". Second, like the models developed by Arnott and Lewis (1979) and Anderson (1986), we are also going to extend this basic model in other to consider differential property taxes before and after the development.

The paper is organized as follows: The following section sets up the basic model of Williams (1991), discusses the most important features of this basic model and extends it in a way to consider time to build and differential property taxes before and after its development. Section 3 provides an empirical analysis of the extended model developed here for the case of the residential real estate market in the city of Rio de Janeiro. Furthermore, these empirical results will be used to verify the compatibility of the theoretical model developed here with the reality of this market. Section 4 shows the implicit risk premiums for the unit cash inflows. The last section is the conclusion.

2. Modeling Optimal Timing and Density of Development.

In recent years there is a growing bibliography on the study of real estate market, making use of real options theory. Shoup

(1970) was probably one of the first papers. Later, Arnott and Lewis (1979) refined the Shoup analysis by accounting explicitly, not only for the timing of the property development, but also its structural density. They also introduce, in the model, differential property taxes before and after development. Other empirical and theoretical papers analyzing options in the real estate market are: Titman (1985), Anderson (1986), Clarke and Reed (1988), Williams (1991), Capozza and Yuming Li (1994), Williams (1997), Grenadier (1999), among others. This paper is based on the Williams (1991) model. This last model has the advantage of being both extremely realistic, taking into account a great number of relevant details, and relatively simple, as a closed-form solution is derived from it.

In this section we will develop Williams's basic model and extend it in a way to include time to build. First, we are going to discuss the assumptions behind the model. Second, we are going to determine the optimal investment rules and discuss some comparative statics. Finally, we will introduce taxes in the analysis and observe how these changes occur.

It is important to outline that some relevant features will not be considered in the present model. First, the model will not take into account the impact of competitive interactions under exogenous and endogenous competition. Related publications regarding the use of Game theory to model strategic behavior in a competitive environment include Granadier (1999). Second, the model presented here also will not take into account options to switch the use of a property (for instance the option to convert a commercial property into an industrial one). This kind of analysis can be found in Capozza and Yuming Li (1994) and Williams (1997).

2.1. Basic Assumptions.

First of all, we suppose that the investment is irreversible

in a sense that building will freeze the land forever in that particular use. In other words, we do not consider option to switch the use of the land. According to Clarke and Reed (1998), cash inflows and costs of construction are not affected by a singular investor optimal decision rule. To keep matters as simple as possible, we will consider that the option to construct never expires and that the building does not depreciate. For the time being, we will suppose that there will not be any property taxes, but this assumption will be relaxed later.

Furthermore, we will suppose that the investor already owns an undeveloped property. The date when the owner acquired his property is denoted by $t = 0$. At any time $\forall t \geq 0$ he can develop his property at some feasible density Q subject to zoning restrictions. Once a building is completely developed it is prohibitively costly to add new units. So, denoting by δ the maximum density permitted under zoning regulations, density Q has to satisfy the following restriction:

$$1 \ll Q \ll \delta \tag{1}$$

Both, the units cash inflows, x_2, and the unit development costs, x_1, evolve stochastically through time and are driven by geometric Wiener processes:

$$dx_i = \mu_i x_i dt + \sigma_i x_i dz_i \quad \text{For} \quad i = 1, 2 \tag{2}$$

Where μ_i is the expected growth rate of x_i, σ_i is the standard deviation per unit of time and dz_i is the increment of a standard Wiener process. Also, the covariance $\sigma_{12} = cov\,(\partial x_1, \partial x_2)$ between the rates of growth in x_1 and x_2 is constant per unit of time. A property developed at the density Q produces the net cash inflow Qx_2 per unit of time

and follows a power cost function[1]:

$$C(Q) = Q^{\gamma} x_1 \tag{3}$$

with $\gamma > 1$ resulting in an increasing convex cost function. As it is highlighted in Titman (1985), the reason for the convexity of the cost function is that, as the number of floors in a building increases, the cost per floor also increases.

To value the undeveloped and developed properties, some additional assumptions are necessary. The riskless rate of interest r is constant per unit of time. Also, the stochastic evolution of both the unit construction cost, x_1, and the unit cash inflow, x_2, can be replicated from portfolios of securities that are traded continuously without transaction costs in the perfectly competitive capital market. For each portfolio, $i = 1, 2$, the excess mean return per unit of standard deviation equals some constant λ_i. So, the risk-adjusted, expected growth rates can be described as follows:

$$v_i \equiv \mu_i - \lambda_i \sigma_i \tag{4}$$

Finally, an additional assumption that the development of a new property takes time to build and that the operating cash inflows must be discounted by a constant factor $0 \leq \Theta \leq 1$, proportional to the value of the developed property, must be considered. This refinement of the basic model will be based on the model of Majd and Pindyck (1987). The reason for the discount in the net cash inflow is that, although a great number of units will be already sold before the beginning of

[1] The model also considers that the undeveloped property has a net cash inflow per unit of time, βx_2, with the constant $1 \geq \beta \geq 0$. This net cash inflow can be originated, for instance, from using the vacant land as a parking lot. Greater details can be found in Medeiros (2001).

the construction, the majority will only be sold during and after the construction is completed. It is also true that the units sold before the beginning of the construction will have their prices affected by the expected time that will be spent in the construction. So, the longer it takes to build a property, the lower the investor's net cash inflow will be.

2.2. Initial Model.

First we will focus on the developed property value. Over time, the value of the developed property evolves in response to the stochastic evolution of its net cash inflow, x_2. So, denoting by $P(x_2)$ the current price of the unit of density of the developed property and after some calculations[2] we obtain the following differential equation:

$$P' v_2 x_2 + \frac{1}{2} P'' \sigma_2^2 x_2^2 + Q x_2 - (\Theta + r) P = 0 \qquad (5)$$

The differential equation (5) must be satisfied for all feasible net cash inflows, $\forall x_2 \geq 0$, and the two boundary conditions[3]:

$$x_2 = 0 \rightarrow P(0) = 0 \qquad (6a)$$

$$P(x_2)/x_2 \ll \xi \quad \text{for} \quad 0 < \xi < \infty \qquad (6b)$$

The solution for the developed property is immediate. The differential equation (5), subject to the boundary conditions (6a) and (6b), has the unique solution:

[2] The calculations for the derivation of the differential equation (5) can be found at the Appendix.

[3] To obtain further information on these boundary conditions, the reader can refer to Medeiros (2001).

$$P(x_2) = \pi_N Q x_2 \tag{7}$$

with the constant,

$$\pi_N = \frac{1}{r + \Theta - v_2} \tag{8}$$

It can be seen that in the case of instantaneous development, that is, $\Theta = 0, \pi_N$ is greater and the value of the developed property, $P(x_2)$, is also greater. So, the inclusion of the time to build in the model leads to a decrease in the value of the developed property.

Over time the value of the undeveloped property is driven by the random evolution of both the unit construction cost and the unit cash inflow. Conditional on the current values, $x = (x_1, x_2)$, the undeveloped property has the value $V(x)$. After some calculations[4] we can also obtain the following differential equation:

$$\begin{aligned} V_1 v_1 x_1 + V_2 v_2 x_2 + \frac{1}{2} V_{22} \sigma_2^2 x_2^2 + \frac{1}{2} V_{11} \sigma_1^2 x_1^2 \\ + V_{12} \sigma_{12} x_1 x_2 + \beta x_2 - rV = 0 \end{aligned} \tag{9}$$

This partial differential equation must be satisfied for all $\forall x_2 \geq 0$ and $\forall x_1 \geq 0$. As a result, the values of the developed and undeveloped properties must satisfy the inequalities $0 \ll V(x) \ll P(x_2)$ and equation (9) is subject to the following conditions[5]:

[4] The calculations for the derivation of the differential equation (9) can be found at the Appendix.

[5] To obtain some information on these conditions, the reader can refer to Medeiros (2001).

$$V(x_1, 0) = 0 \tag{10a}$$

$$V(x^*) = P(x_2^*) - Q^{*\gamma} x_1^* \tag{10b}$$

$$Q^* = \arg\max_{Q} \{(P{x_2}^*) - Q^{\gamma}{x_1}^* : 1 \ll Q \ll \delta\} \tag{10c}$$

and also to the Smooth-Pasting condition:

$$V_1(x^*) = -Q^{*\gamma} \quad \text{and} \quad V_2(x^*) = P'(x_2^*) \tag{10d}$$

Equation (10b) relates the value of the developed and undeveloped property. It says that if the property is developed at the optimal timing and density ($x = x^* \quad \text{and} \quad Q = Q^*$), then the value of the undeveloped property must be equal to the price of the developed property $P(x_2^*)$ minus the cost of development. In addition, as stated in equation (10c) and outlined by Titman (1985), the investor determines the optimal density by maximizing his profit function subject to the zoning restrictions.

In addition, a ratio must be defined in the following way: the unit cash inflow is divided by the unit construction cost, $y = x_2/x_1$. Also, two parameters will be defined as follows:

$$\eta \equiv -\left(\frac{v_2 - v_1}{\omega^2} - \frac{1}{2}\right) + \sqrt{\left(\frac{v_2 - v_1}{\omega^2} - \frac{1}{2}\right)^2 + 2\frac{r - v_1}{\omega^2}} \tag{11}$$

$$\psi \equiv \frac{\eta}{\eta - 1} \tag{12}$$

with,

$$\omega^2 = \sigma_1^2 + 2\sigma_{12} + \sigma_2^2 \tag{13}$$

After this transformation[6] we can conclude that the solution to the partial differential equation (9) subject to the restrictions (10a,b,c,d) and also to the value of the developed property (7)-(8) has the solution specified below:

$$y^* = \begin{cases} \left(\frac{1}{1-\beta\psi}\right)\frac{\psi}{\pi_N}, & \frac{\psi}{1-\beta\psi} \leq \gamma \\ \frac{\gamma}{\pi_N}\left(\frac{\beta\gamma\psi}{\gamma-\psi}\right)^{\gamma-1}, & \frac{\delta\psi}{\delta-\beta\psi} \leq \gamma \leq \frac{\psi}{1-\beta\psi} \\ \frac{\psi}{\pi_N}\left(\frac{\delta^\gamma}{\delta-\beta\psi}\right), & \gamma < \frac{\delta\psi}{\delta-\beta\psi} \end{cases} \tag{14}$$

$$Q^* = \begin{cases} 1, & \frac{\psi}{1-\beta\psi} \leq \gamma \\ \frac{\beta\gamma\psi}{\gamma-\psi}, & \frac{\delta\psi}{\delta-\beta\psi} \leq \gamma \leq \frac{\psi}{1-\beta\psi} \\ \delta, & \gamma < \frac{\delta\psi}{\delta-\beta\psi} \end{cases} \tag{15}$$

So, the owner of an undeveloped property will optimally construct a building with density Q^* when the ratio $y = x_2/x_1$ equals the critical value y^*. In addition, the longer it takes to build the higher the critical value y^*, in other words, the ratio of the unit cash inflow relative to the unit construction cost will increase while Q^* remains the same. So, in general, properties take longer to be developed while the optimal density remains the same. The explanation for this result is that the longer the time spent in the construction process, the higher the discount in the net cash inflow and, to compensate for this discount, the developer will ask for a higher cash inflow. All the comparative

[6] The calculations of the transformations can be found in Medeiros (2001).

statics including the time to build are summarized in the next table.

Increase in	y^*	Q^*	Restrictions on Parameters[7]
γ	Inconclusive	Decrease	
δ	Decrease	Increase	$\gamma < \frac{\delta\psi}{\delta-\beta\psi}$
σ_1^2 σ_2^2	Increase	Increase	$v_1 \geq v_2$
$\text{Cov}(x_1, x_2)$	Decrease	Decrease	$v_1 \geq v_2$
R	Decrease	Decrease	$v_1 \geq v_2$
Θ	Increase	Unchanged	$v_1 \geq v_2$

2.3. Including Taxes.

The model developed so far can be easily extended to include taxes. We will base our extension on the model developed by Arnott and Lewis (1979). Denote property tax rate before development as τ_a and property tax rate after development as τ_d. As outlined by Anderson (1986) permitting different tax rates before and after development may be justified in two ways. First, it is possible that the municipality will set differential tax rates on the property before and after development. This, however, is unlikely to happen since most units of local government do not discriminate taxes in this way. What is more likely is that the assessments may differ in fundamental ways before and after development, making the effective tax rates differ even though the nominal rates are identical.

[7]In most cases the results are true only when $v_1 \geq v_2$. A reasonable explanation for this condition is that, over long periods, construction costs affect the aggregate supply of rentable space, which, in turn, largely determines rents and thereby operating cash inflows. So, over the long run, rents have a higher but not a lower limit set by construction costs. Thus, the expected growth rate of cash inflows, μ_2, is limited above by the expected growth rate of construction costs, μ_1, while the variance σ_2^2 is bounded below by σ_1^2.

First, we note that the value of the developed property $P(x_2)$, is affected only by the property tax rate after development, τ_d. These taxes can be seen as a discount proportional to the value of the developed property ($\tau_d P$). Once again, after some calculations[8], we obtain the following differential equation:

$$P' v_2 x_2 + \frac{1}{2} P'' \sigma_2^2 x_2^2 + Q x_2 - (\Theta + \tau_d + r) P = 0 \qquad (16)$$

The differential equation (16) subject to the boundary conditions (6a) and (6b) has the unique solution:

$$P(x_2) = \pi_d Q x_2 \qquad (17)$$

where,

$$\pi_d = \frac{1}{r + \Theta + \tau_d - v_2} \qquad (18)$$

We can conclude that the value of the developed property decreases when we include property taxes after development. In addition, although τ_d does not affect the value of the undeveloped property directly, because the value of the undeveloped property is related to the value of the developed property, it affects $V(x)$ indirectly. We can also see that the higher the tax rates τ_d the smaller the value of the undeveloped property. According to the analysis done before, we can obtain the same equations for the critical values y^* and Q^*, only substituting the old parameter, π_N, for the new parameter, π_d, noting that $\pi_N \geq \pi_d$. So, for positive values of tax rates after development ($\tau_d > 0$) the critical value y^* is now higher than before the time

[8] The calculations for the derivation of the differential equation (16) can be found at the Appendix.

when the taxes were null. The optimal density, Q^*, remains the same after the inclusion of taxes. So, obtaining the same conclusions as Arnott and Lewis presented before, an increase in the property tax rate after development delays the construction decision while it maintains the same optimal density.

Second, we note that the value of the undeveloped property is directly affected by the property tax rate before development, τ_a. These taxes can be seen as a discount proportional to the value of the undeveloped property ($\tau_a V$). Once again, after some simple calculations[9], we obtain the following differential equation:

$$V_1 v_1 x_1 + V_2 v_2 x_2 + \frac{1}{2} V_{22} \sigma_2^2 x_2^2 + \frac{1}{2} V_{11} \sigma_1^2 x_1^2 + V_{12} \sigma_{12} x_1 x_2 + \beta x_2 - \tau_a V - rV = 0 \tag{19}$$

Once again, defining the ratio $y = x_2/x_1$, after some transformations and defining the two new parameters:

$$\vartheta = -\left(\frac{v_2 - v_1}{\omega^2} - \frac{1}{2}\right) + \sqrt{\left(\frac{v_2 - v_1}{\omega^2} - \frac{1}{2}\right)^2 + 2\left(\frac{r + \tau_a - v_1}{\omega^2}\right)} \tag{20}$$

$$\chi \equiv \frac{\vartheta}{\vartheta - 1} \tag{21}$$

We can conclude that the differential equation (19) subject to conditions 10a-d and to the value of the developed property 7-8 has the solution specified below:

$$y^{**} = \begin{cases} \left(\frac{1}{1-\beta\chi}\right)\frac{\chi}{\pi_N}, & \frac{\chi}{1-\beta\chi} \leq \gamma \\ \frac{\gamma}{\pi_N}\left(\frac{\beta\gamma\chi}{\gamma-\chi}\right)^{\gamma-1}, & \frac{\delta\chi}{\delta-\beta\chi} \leq \gamma \leq \frac{\chi}{1-\beta\chi} \\ \frac{\chi}{\pi_N}\left(\frac{\delta^\gamma}{\delta-\beta\chi}\right), & \gamma < \frac{\delta\chi}{\delta-\beta\chi} \end{cases} \tag{22}$$

[9] The calculations for the derivation of the differential equation (19) can be found at the Appendix.

$$Q^{**} = \begin{cases} 1, & \frac{\chi}{1-\beta\chi} \leq \gamma \\ \frac{\gamma\beta\chi}{\gamma-\chi}, & \frac{\delta\chi}{\delta-\beta\chi} \leq \gamma \leq \frac{\chi}{1-\beta\chi} \\ \delta, & \gamma < \frac{\delta\chi}{\delta-\beta\chi} \end{cases} \tag{23}$$

We can see that for $v_1 \geq v_2$ an increase in the property tax rate before development leads to a decrease in both critical values, that is, $Q^* > Q^{**}$ and $y^* > y^{**}$. So, obtaining the same conclusions as Arnott and Lewis presented before, an increase in the tax rate before development leads not only to a decrease in the optimal density, but also to a increase in the speed of the construction process, once a smaller y^* is necessary to begin a new development.

In short, we can conclude that uncertainty plays an important role in the main decisions of real estate market. We can also note that not only the inclusion of the time to build, but also taxes substantially affect the results of the model. However, an empirical analysis is crucial for us to have a real dimension of how the theoretical model developed here behaves in practice. So, in the following section, we will apply the model developed so far to the case of the residential real estate market in the city of Rio de Janeiro.

3. Empirical Analysis of the Residential Real Estate Market in the City of Rio de Janeiro.

3.1. Introduction.

Now we will develop an empirical analysis of the model developed so far in the case of the residential real estate market in the city of Rio de Janeiro. Each parameter will be estimated and, for those parameters that we can not estimate, we will try to find the best proxy. However, one of the objectives of the present paper is to verify the compatibility of the theoretical model developed here with the reality of this market. For

this purpose, we will have to estimate the real business cycle of the real estate market in the city of Rio de Janeiro. Knowing the periods of high activity in the real estate market, we can compare this with the periods of high activity based on the theoretical model and then, observe if both periods are compatible. So, we will estimate the real business cycle of the real estate market in the city of Rio de Janeiro. Then we will present the main results of the model and confirm the compatibility of the theoretical model with the reality of the residential real estate market.

3.2. Estimating the Geometric Brownian Motion.

In what follows we will estimate from real data the parameters that characterize the Geometric Brownian Motion described earlier in equation (2) that drives both the net cash inflow and the construction costs. In this procedure we are implicit assuming that the data from both the net cash inflow and the construction costs follows a lognormal distribution without testing. Although there are few works that address the empirical implications of option-based models for real assets, we can find some related works such Quigg (1993)[10], Paddock, Stiegel, and Smith (1988)[11] among others that follows somewhat similar procedures.

[10] Quigg (1993) assume that both the development costs and the price of the underlying asset, the building, evolve as Geometric Brownian motions. Using a data set that consists of a large number of real estate transactions within the city of Seattle and indexes of per-square-foot construction costs for various types and qualities of buildings, the paper proceed with the empirical results, estimating the implied standard deviations of individual commercial real estate prices, without reporting a formal test for the assumption of log normality.

[11] Paddock et al. (1988) uses option valuation theory to develop a new approach to valuing leases for offshore petroleum. They also assume that the statistical distribution of oil and gas reserves for a given tract is joint lognormal without reporting a formal test.

A) Parameters of the Real Estate Selling Prices Process

We already know that a Geometric Brownian Motion described in equation (2) drives both the net cash inflow and the construction costs. So, considering as a net cash inflow, x_2, the income from selling the property's units, to estimate both the variance and the expected growth rate of the net cash inflow, μ_2 and σ_2^2, we will use monthly data on the residential real estate selling prices in the city of Rio de Janeiro published by SECOVI-RJ[12]. These data begin in July 1994 and end in July 2000. These values were calculated using the arithmetic average of all supply selling prices per location and type of apartment in a specific month of the year published by the O Globo newspaper. We must outline the fact that these data do not have any kind of equalization, giving the same weights for units of different ages, locations and floor. As a consequence, in some months there is a variation above the regular one, that does not reflect the true variation in the real estate selling prices. So, in other to make an accurate estimate of the parameters, we must pay attention to these distorted findings.

We will use as an estimate of the net cash inflow volatility, σ_2^2, the unbiased estimator of σ. The estimates will be calculated after taking the natural logarithm of the relative prices. Tables 1A-1B show the estimates of the expected growth rates of the selling prices, μ_2,[13] and the volatility, σ_2^2, for the various locations and types of buildings.

[12] SECOVI-RJ stands for Union of Buying Firms, Sales, Rental and Administration of estate in the sites in the city of Rio de Janeiro.

[13] We must remember that after taking the natural logarithm of the relative prices, by Ito's Lemma, we can conclude that the expectation of the monthly growth rates of the real estate selling prices can be described as: $\bar{\mu}_2 = \hat{\mu}_2 + \frac{\hat{\sigma}_2^2}{2}$

Table 1-A

Locations	Expected Growth Rate of the Unit Cas Inflow: μ_2						
	Studio	1b.	2b.	3b.	4b.	average	Annual average
Andarai/Grajau	1.51%	1.49%	1.28%	1.21%	1.23%	0.88%	11.04%
Bangu/Campo G.	1.58%	1.54%	1.62%	1.57%	1.59%	1.56%	20.46%
Barra/Recreio	1.41%	1.49%	1.40%	1.27%	1.33%	1.15%	14.69%
Botafogo/Humaita	1.30%	1.26%	1.55%	1.33%	1.22%	1.17%	15.04%
Cascadura/Piedade	1.54%	1.53%	1.51%	1.52%	1.50%	1.52%	19.80%
Centro	1.55%	1.27%	1.40%	1.37%	1.52%	1.21%	15.47%
Copacabana/Leme	1.49%	1.31%	1.22%	1.20%	1.33%	1.17%	15.04%
Flamengo/Catete	1.54%	1.26%	1.44%	1.34%	1.42%	1.33%	17.24%
Gavea	1.47%	1.45%	1.60%	1.51%	1.20%	1.21%	15.56%
Ilha	1.49%	1.47%	1.17%	1.56%	1.30%	1.29%	16.68%
Ipanema	1.39%	1.62%	1.20%	1.40%	1.18%	1.17%	14.93%
Iraja/Vista A.	1.63%	1.62%	1.61%	1.60%	1.59%	1.61%	21.13%
Jacarepagua	1.52%	1.61%	1.37%	1.27%	1.37%	1.19%	15.26%
J.Botanico	1.20%	1.18%	1.15%	1.37%	0.96%	1.13%	14.38%
Lagoa	1.35%	1.33%	1.34%	1.60%	1.32%	1.30%	16.78%
Leblon	1.44%	1.42%	1.42%	1.40%	1.36%	1.41%	18.28%
Laranjeiras/C.Velho	1.23%	1.20%	1.19%	1.48%	1.08%	1.19%	15.27%
Madureira	1.40%	1.38%	1.58%	1.40%	1.55%	1.36%	17.64%
Meier/Lins	1.12%	1.20%	1.47%	1.36%	0.90%	1.05%	13.38%
Ramos/Leopoldina	1.62%	1.60%	1.61%	1.59%	1.58%	1.60%	20.93%
S.Teresa/Gloria	1.44%	1.43%	1.46%	1.45%	1.44%	1.46%	18.94%
S.Cristovão/Benfica	1.26%	1.24%	1.30%	1.19%	1.22%	1.22%	15.65%
Tijuca/Rio C.	1.33%	1.02%	1.41%	1.57%	1.26%	1.11%	14.17%
Urca	1.37%	1.34%	1.35%	1.67%	1.40%	1.31%	16.95%

Note: 1b.=one room (bedroom) apartment; 2b.=two rooms (bedrooms) apartment; 3b.=three rooms (bedrooms) apartment; 4b.=four rooms (bedrooms) apartment.

Table 1-B

Locations	Standard Deviation of the Growth Rates of the Unit Cash Inflow: σ_2					
	Studio	1b.	2b.	3b.	4b.	average
Andarai/Grajau	12.69%	10.82%	8.79%	9.71%	13.03%	6.40%
Bangu/Campo G.	16.15%	14.25%	13.76%	10.50%	13.49%	8.99%
Barra/Recreio	13.03%	9.30%	11.78%	8.32%	9.05%	5.78%
Botafogo/Humaita	13.39%	12.10%	9.86%	8.86%	11.10%	7.21%
Cascadura/Piedade	15.53%	15.00%	10.38%	8.14%	11.04%	7.67%
Centro	8.35%	9.05%	8.24%	9.67%	7.52%	5.29%
Copacabana/Leme	6.74%	6.54%	8.63%	9.21%	9.03%	5.48%
Flamengo/Catete	7.27%	7.71%	11.12%	10.91%	7.8%7	5.78%
Gavea	13.32%	11.87%	9.29%	8.70%	7.26%	5.63%
Ilha	11.49%	13.09%	9.96%	11.72%	12.56%	8.50%
Ipanema	11.83%	11.38%	10.13%	9.41%	5.84%	5.37%
Iraja/Vista A.	13.72%	13.72%	13.77%	12.32%	11.27%	8.74%
Jacarepagua	9.91%	11.27%	9.07%	9.21%	10.41%	6.01%
J.Botanico	11.88%	10.97%	9.53%	8.02%	7.25%	5.14%
Lagoa	11.05%	11.87%	9.64%	8.58%	8.09%	5.75%
Leblon	11.46%	12.34%	10.71%	10.09%	7.10%	5.69%
Laranjeiras/C.Velho	10.11%	8.86%	12.32%	9.07%	8.26%	5.94%
Madureira	14.06%	13.46%	10.44%	9.76%	9.88%	7.14%
Meier/Lins	11.77%	8.70%	10.89%	9.25%	7.20%	5.41%
Ramos/Leopoldina	14.66%	13.46%	11.97%	12.34%	11.55%	8.80%
S.Teresa/Gloria	8.49%	8.90%	6.11%	9.91%	13.00%	7.50%
S.Cristovão/Benfica	13.91%	14.21%	12.20%	10.86%	11.34%	7.90%
Tijuca/Rio C.	10.45%	7.18%	8.72%	10.55%	9.38%	6.56%
Urca	9.25%	9.64%	7.87%	9.12%	9.53%	5.96%

Note: 1b.=one room (bedroom) apartment; 2b.=two rooms (bedrooms) apartment; 3b.=three rooms (bedrooms) apartment; 4b.=four rooms (bedrooms) apartment.

We can note the existence of a similar behavior between the real estate selling prices and its variance per location. As can be seen in figure I, the smaller the selling prices for a specific location, the higher the respective variances. So, in average, locations where the buildings are cheaper, the properties have a greater variance than locations where the buildings are more expensive. We must be cautious not to draw any definite conclusion using only the present data. We must also emphasize that even if the last conclusion is true, it can be a particular characteristic of the residential real estate market in the city of Rio de Janeiro not a general rule. Even more, looking again at table I one can see that some locations have their particular behavior. For instance, Meier and Centro are locations where the expected growth rates are higher, but the variances are smaller. A reasonable explanation is the way these data are organized (without any equalization). For instance, it can be true that the supply of units in this location is more homogeneous than in other locations leading us to a smaller volatility in the parameter prices.

B) Parameters of the Construction Costs Process

To estimate the parameters that characterize the construction costs, μ_1 and σ_1^2, we will use the average monthly values of the mean return unitary cost per square meter of construction (Mean CUB-RJ) for the city of Rio de Janeiro. The Sinduscon/RJ publishes the CUB-RJ[14]. The data used begins in July 1994 and ends in January 2001. Once again, we will calculate the variance and the expected return of the natural logarithm of the relative price over the period specified. Table 2 below shows the values of the parameters, μ_1 and σ_1^2, estimated using data on CUB-RJ.

Table 2

Index	Estimatives	
	μ_1	Standard Deviation
CUB-RJ	0.74%	2.32%

C) Risk Premium

For the time being we will establish the risk premium of the construction costs around 1%, $\lambda_1 = 0.01$. Then, we will obtain the implicit risk premium for the net cash inflow, λ_2, for the various critical dates and zones as will be described later in section 4.

D) Covariance

We have verified that the values of the estimates of the covariance, σ_{12}, were very small, nearly zero, for monthly comparisons. One reasonable explanation is that it takes some time for the selling prices to respond to a variation in the construction cost. So, to keep matters as simple as possible, we will suppose that the covariance is null.

[14] Sinduscon/RJ stands for Union of Civil Construction Industry for the city of Rio de Janeiro.

3.3. Estimating the Remaining Parameters.

A) Risk Free Return (r)

As a proxy for the risk free return, we will consider SELIC's (published by the Brazilian Central Bank) monthly variation annualized average for the last two years (1999 and 2000). The value of this proxy is around 21.59%. The reason for limiting the average only for the last two years is that if we used an average over a longer period we would forecast an extremely high risk free return tax, not compatible with the economic stabilization trend in Brazil.

B) Legal Limitations (δ)

As a proxy for the legal limitations we will consider the IAA index (percentage of the land used for construction). The city government of Rio de Janeiro in 1992 publishes the IAA index. We will also consider the regulations for the smallest site size (min site) that can be used for residential construction. Table 3 shows the IAA's and smallest site size for the various locations in the city of Rio de Janeiro[15]. In the last five columns of this table we can observe the maximum number of units that can be developed for each type of building[16].

[15] Some assumptions were made for locations that do not have a specific IAA or smallest site size regulation.

[16] The model developed here considers a cash inflow originated from the use of undeveloped property described by the parameter β. As a proxy for this parameter we consider the parking lot rent cash inflow. The values of the β's for the various locations and types of apartments have been estimated. These values can be found in Medeiros (2001).

Table 3: Determining Legal Limitations

Locations	IAA	Min Site	δ				
			Studio	1B.	2B.	3B.	4B.
			($40m^2$)	($50m^2$)	($75m^2$)	($105m^2$)	($145m^2$)
Andarai	4.0	600	60	48	32	23	16
Grajau	3.0	600	45	36	24	17	12
Bangu/C.Grande	3.5	360	32	25	17	12	8
Barra/Recreio	3.5	600	53	42	28	20	14
Botafogo/Humaita	3.5	600	53	42	28	20	14
Cascadura/Piedade	3.0	360	27	22	14	10	7
Centro	5.0	600	75	60	40	29	20
Copacabana/Leme	3.5	600	53	42	28	20	14
Flamengo/Catete	3.5	600	53	42	28	20	14
Gavea	3.5	600	53	42	28	20	14
Ilha	1.5	360	14	11	7	5	4
Ipanema	3.5	600	53	42	28	20	14
Iraja/Vista A.	3.0	360	27	22	14	10	7
Jacarepagua	3.5	360	32	25	17	12	8
J.Botanico	3.5	600	53	42	28	20	14
Lagoa	3.5	600	53	42	28	20	14
Leblon	3.5	600	53	42	28	20	14
Laranjeiras/C.Velho	3.5	600	53	42	28	20	14
Madureira	4.0	360	36	29	19	14	10
Meier/Lins	3.5	360	32	25	17	12	8
Ramos/Leopoldina	3.0	360	27	22	14	10	7
S.Teresa	1.0	360	9	7	5	3	2
Gloria	3.5	600	53	42	28	20	14
S.Cristovão/Benfica	5.5	360	50	40	26	19	13
Tijuca	3.5	360	32	25	17	12	8
Rio Comprido	2.5	360	23	18	12	9	6
Urca	1.0	600	15	12	8	6	4

Note: 1b.=one room (bedroom) apartment; 2b.=two rooms (bedrooms) apartment; 3b.=three rooms (bedrooms) apartment; 4b.=four rooms (bedrooms) apartment.

C) Development Cost of Density (γ) *and Time to Build*(Θ)

For the time being, we will consider an instantaneous development, that is, $\Theta = 0$, and $\gamma = 1.2$. Later we will attribute higher values for these parameters.

3.4. Estimating the Real Business Cycle of the Real Estate Market.

Before analyzing the development of the model and its respective results, we must discuss the estimation of the real business cycles of the real estate market in the city of Rio de Janeiro. The importance of estimating the real estate real business cycles is that we can obtain the periods of cycle reversion, that is, the periods when the real estate market start to increase, that could be understood in the present discussion as the time to begin a new property development. Knowing these periods, the next step is to estimate the evolution of the y's (the ratios of the unit cash inflows divided by the unit construction costs) for the various locations along the time. Then, using the critical values, y^*'s, calculated from the theoretical model, we can obtain the optimal dates for construction searching for the periods along the evolution of the y's where this last ratio reach its respective critical value y^*. This would give us the optimal dates for construction according to the theoretical model. Finally we can check if these optimal dates are compatible with the periods of cycle reversion described by the real estate cycles.

We will estimate the real business cycles of the real estate market using monthly data on the proportion of the occupied population per field of activity and metropolitan region according to PME (Monthly Employment Research) published by IBGE. These data begin in January 1990 and end in December 2000 and show the relation between the number of people working in a specific field of activity (for instance, Civil

Construction, Services, Transformation Industry, etc.) and the total number of working people in a specific period. We will focus on the proportion of people working in the civil construction in the city of Rio de Janeiro. However, we must note that this particular analysis does not capture the actual absolute variation of the population working in civil construction since we can see periods where the relation between the number of people working in the civil construction and the total number of working people can decrease not because there was a reduction in this last sector but mainly because there was a higher growth in another sector in the same period. As a solution, we will use the monthly variation of the absolute amount of people working in the civil construction in the city of Rio de Janeiro. This is a good proxy for the evolution of real business cycles of the real estate market since the period when a greater number of people are working in this sector is the same as when this market is working with higher activity. However we must keep in mind that this proxy has some limitations since the civil construction activity include not only work in the construction of residential buildings but also work in the construction of commercial, industrial and public buildings. Figure 2 shows the twelve months moving average evolution of the civil construction market based on the monthly variation of the absolute amount of people working in this activity in the city of Rio de Janeiro that is our proxy for the real business cycle of the real estate market. Observing this graph we can see that January 1996, January 1997, July 1998 and September 1999 are approximately periods of cycle reversion. We will then check if these periods are compatible with the optimal dates for beginning a new building development according to the theoretical model. However we must outline the fact that this comparison is valid only approximately, it is not completely precise, since the proxy used here for describing the real business cycles has

the limitations specified before.

3.5. Model Development and Main Results.

Before analyzing the results, some important considerations must be made. Because density and unit of time measure both the unit cash inflow and the construction costs, we have to annualize the cash inflow. For this reason we will suppose that the buildings were sold with a ten-year loan while the cost of construction is entirely paid in the first year of construction. So, to annualize the cash inflow we will consider 10% of the total property selling price. However, the results will be presented in terms of total selling value in relation to the total construction cost value. Furthermore, to measure the net cash inflow and construction costs per unit of density, we will make the following calculations:

Unit cash inflow per unit of density = (unit selling price) ÷ (average unit size in square meter) × (min site in square meter)

Construction cost per unit of density = (Construction cost per square meter) × (min site in square meter)

Paying attention to these initial considerations, we will now consider the main results of the basic model without taxes and, finally, we will observe how the results vary when we include taxes.

3.5.1 Basic Model.

A) Mean Results (Table 4)

To calculate the values presented in table 4 we will use the mean parameters (the mean standard deviation and the mean expected growth rate) that describe the net cash inflow and the construction costs[17] for all locations. We will also use the mean β's per location, the risk premium that will be specified in section 4, and the remaining parameters as described in the table below.

$$r = 21.59\%;\ \gamma = 1.2;\ \Theta = 0;\ \sigma_{12} = 0;\ \lambda_1 = 1\%;\ \delta = IAA.$$

First, comparing the results for each location in this table, we observe that Santa Teresa has a very high critical value y^*. One reasonable explanation for this result is that in this specific location the legal limitations are very severe, $\delta = 1.0$, leading to worse conditions of construction since the initial value considered for the cost of construction of an additional unity is low,

[17] These mean values were calculated using the aggregate value for all selling prices of a specific location giving equal measures for each type of building.

$\gamma = 1.2$, leading to a situation where profitability increases according to a higher number of units constructed. This conclusion can be confirmed by the analysis of the critical values for the optimal density, Q^*, that are equal to the legal limitation in all locations, that is, $Q^* = \delta$. However, Urca is a location where the legal limitations are also severe, $\delta = 1.0$, but it does not have a high critical value y^*. So, the difference between these two locations is that the mean volatility for the unit cash inflow is higher in Santa Teresa than in Urca. We can also conclude that the locations where the mean volatility for the unit cash inflow is higher, the corresponding critical values necessary to begin a new development are also higher, in other words, the speed of the construction process is slower because of the higher uncertainty.

B) Types of Building (Table 5)

For the calculations of the values presented in the table 5 we used the same parameters specified in the previous calculations of the mean results of table 4 only changing the mean estimates of standard deviations and growth rates for the specific values of each type of construction. We can conclude that the critical values for each type of building are now higher than the mean values presented before. One reasonable explanation is that the individual variances are higher than the mean ones, turning the new critical values higher too.

Table 4: Mean Results

Locations	Critical Values	
	Q^*	y^*
Andarai	4.0	179.5%
Grajau	3.0	172.1%
Bangu/C.Grande	3.5	200.4%
Barra/Recreio	3.5	161.8%
Botafogo/Humaita	3.5	169.5%
Cascadura/Piedade	3.0	192.0%
Centro	5.0	174.9%
Copacabana/Leme	3.5	164.9%
Flamengo/Catete	3.5	167.7%
Gavea	3.5	163.8%
Ilha	1.5	168.0%
Ipanema	3.5	163.2%
Iraja/Vista A.	3.0	226.0%
Jacarepagua	3.5	166.5%
J.Botanico	3.5	162.8%
Lagoa	3.5	163.6%
Leblon	3.5	164.9%
Laranjeiras/C.Velho	3.5	162.8%
Madureira	4.0	176.4%
Meier/Lins	3.5	167.7%
Ramos/Leopoldina	3.0	214.0%
S.Teresa	1.0	317.0%
Gloria	3.5	171.0%
S.Cristovão/Benfica	5.5	182.6%
Tijuca	3.5	171.0%
Rio Comprido	2.5	159.7%
Urca	1.0	165.0%

Table 5: Types of Apartments

Locations	Critical Values									
	Studio		1B.		2B.		3B.		4B.	
	Q^*	y^*	Q^*	y^*	Q^*	y^*	Q^*	y^*	Q^*	y^*
Andarai	4.0	187.0%	4.0	185.3%	4.0	177.0%	4.0	179.0%	1.8	189.0%
Grajau	3.0	190.0%	3.0	190.0%	3.0	175.0%	3.0	175.0%	3.0	185.0%
Bangu/C.Grande	3.5	256.0%	3.5	216.0%	3.5	246.0%	3.5	224.0%	3.5	217.0%
Barra/Recreio	3.5	178.0%	3.5	170.0%	3.5	174.0%	3.5	166.0%	3.5	167.0%
Botafogo/Humaita	3.5	210.0%	3.5	185.0%	3.5	205.0%	3.5	175.0%	3.5	180.0%
Cascadura/Piedade	3.0	229.0%	3.0	217.0%	3.0	210.0%	3.0	211.0%	3.0	197.0%
Centro	5.0	198.0%	5.0	183.0%	5.0	183.0%	5.0	185.0%	5.0	190.0%
Copacabana/Leme	3.5	202.0%	3.5	174.0%	3.5	173.0%	3.5	175.0%	3.5	172.0%
Flamengo/Catete	3.5	237.0%	3.5	175.0%	3.5	189.0%	3.5	180.0%	3.5	173.0%
Gavea	3.5	195.0%	3.5	179.0%	3.5	205.0%	3.5	180.0%	3.5	168.0%
Ilha	1.5	192.0%	1.5	220.0%	1.5	190.0%	1.5	195.0%	1.5	183.0%
Ipanema	3.5	182.0%	3.5	223.0%	3.5	176.0%	3.5	173.0%	3.5	165.0%
Iraja/Vista A.	3.0	331.0%	3.0	258.0%	3.0	227.0%	3.0	232.0%	3.0	215.0%
Jacarepagua	3.5	194.0%	3.5	251.0%	3.5	177.0%	3.5	177.0%	3.5	179.0%
J.Botanico	3.5	184.0%	3.5	176.0%	3.5	174.0%	3.5	171.0%	3.5	172.0%
Lagoa	3.5	178.0%	3.5	177.0%	3.5	172.0%	3.5	201.0%	3.5	168.0%
Leblon	3.5	185.0%	3.5	179.0%	3.5	175.0%	3.5	173.0%	3.5	167.0%
Laranjeiras/C.Velho	3.5	177.0%	3.5	170.0%	3.5	180.0%	3.5	173.0%	3.5	170.0%
Madureira	4.0	197.0%	4.0	192.0%	4.0	229.0%	4.0	184.0%	4.0	204.0%
Meier/Lins	3.5	188.0%	3.5	177.0%	3.5	190.0%	3.5	183.0%	3.5	180.0%
Ramos/Leopoldina	3.0	308.0%	3.0	238.0%	3.0	249.0%	3.0	216.0%	3.0	213.0%
S.Teresa	1.0	510.0%	1.0	386.0%	1.0	488.0%	1.0	425.0%	1.0	376.0%
Gloria	3.5	177.0%	3.5	175.0%	3.5	174.0%	3.5	178.0%	3.5	184.0%
S.Cristovão/Benfica	5.5	201.0%	5.5	203.0%	5.5	193.0%	5.5	210.0%	5.5	196.0%
Tijuca	3.5	190.0%	3.5	177.0%	3.5	191.0%	3.5	255.0%	3.5	178.0%
Rio Comprido	2.5	177.0%	2.5	166.0%	2.5	178.0%	2.5	236.0%	2.5	167.0%
Urca	1.0	275.0%	1.0	220.0%	1.0	186.0%	1.0	180.0%	1.0	202.0%

Note: 1b.=one room (bedroom) apartment; 2b.=two rooms (bedrooms) apartment; 3b.=three rooms (bedrooms) apartment; 4b.=four rooms (bedrooms) apartment.

C) Including Time to Build (Table 6)

Up to now we considered development instantaneous, that is, $\Theta = 0$. However, we know that in practice a property takes time to build. So, table 6 below shows how the mean results demonstrated before change when we take into account a discount of 5%, that is, $\Theta = 0.05$ (remembering that $ln\,(1 + 0.05) \cong 0.05$). The values for the other parameters will remain the same as the ones used in table 4. We can conclude that optimal density remains the same, but the properties will take longer to be developed (y^* are higher now)[18].

Table 6: Including Time to Build θ=5%

Locations	Critical Values	
	Q^*	y^*
Andarai	4.0	261.0%
Grajau	3.0	250.0%
Bangu/C.Grande	3.5	330.0%
Barra/Recreio	3.5	273.0%
Botafogo/Humaita	3.5	291.0%
Cascadura/Piedade	3.0	396.0%
Centro	5.0	308.0%
Copacabana/Leme	3.5	285.0%
Flamengo/Catete	3.5	297.0%
Gavea	3.5	292.0%
Ilha	1.5	285.0%
Ipanema	3.5	280.0%
Iraja/Vista A.	3.0	420.0%
Jacarepagua	3.5	291.0%
J.Botanico	3.5	271.0%
Lagoa	3.5	322.0%
Leblon	3.5	389.0%
Laranjeiras/C.Velho	3.5	285.0%
Madureira	4.0	369.0%
Meier/Lins	3.5	266.0%
Ramos/Leopoldina	3.0	360.0%
S.Teresa	1.0	560.0%
Gloria	3.5	351.0%
S.Cristovão/Benfica	5.5	321.0%
Tijuca	3.5	281.0%
Rio Comprido	2.5	262.0%
Urca	1.0	329.0%

[18] Other parameters analysis such as variations in the risk premium of the construction cost, legal limitations, etc. can be obtained in Medeiros (2001).

3.5.2 Including Taxes.

A) Including Property Taxes before Construction
(*Table* 7)

We will consider as a proxy for property taxes before development, some costs such as site incorporation, IPTU (a Brazilian real estate tax), etc. that are nearly 5.5% of the total value of the vacant land, that is, $\tau_a = 0.055$. If you use equations 22-23 presented in section 2.3 and the values of the other parameters the same way the ones used in table 4, we can obtain the mean results for the various locations including $\tau_a = 5.5\%$ (table 7). We can conclude that an increase in the property taxes before development lead not only to a decrease in the optimal density (for instance Andaraí), but also to an increase in the speed of the construction process since now the y^* necessary to begin a new development is smaller in all locations.

B) Including Property Taxes after Construction
(*Table* 8)

We will consider as a proxy for property taxes after development some taxes that, in practice, are paid during the construction process, such as PIS/Cofins (that are nearly 3.65% of the value of the developed property), CPMF (that is nearly 0.54% of the value of the developed property) and incorporation costs (that are nearly 2.0% of the developed property) with an approximate total value of $\tau_d = 6.20\%$[19]. Using the same values for the other parameters used in table 4, we can conclude that the inclusion of property taxes after development lead to a decrease in the speed of construction process while the optimal density remains the same.

[19] We did not include all the taxes and costs involved in the construction of a building. However, the main objective of this paper is to investigate how fiscal policies influence the real estate market based on the present model and not to establish a detailed study of taxes.

Table 7: Property Taxes Before Construction

Locations	Critical Values			
	$\tau a = 5.5\%$		$\tau a = 11.0\%$	
	Q^*	y^*	Q^*	y^*
Andarai	2.7	160.0%	2.7	157.0%
Grajau	3.0	164.0%	3.0	160.0%
Bangu/C.Grande	3.5	164.0%	3.5	158.0%
Barra/Recreio	3.5	134.0%	3.5	123.0%
Botafogo/Humaita	3.5	137.0%	3.5	124.0%
Cascadura/Piedade	3.0	176.0%	3.0	168.0%
Centro	5.0	138.0%	5.0	124.0%
Copacabana/Leme	3.5	132.0%	3.5	120.0%
Flamengo/Catete	3.5	110.0%	3.5	101.0%
Gavea	3.5	128.0%	3.5	114.0%
Ilha	1.5	114.0%	1.5	103.0%
Ipanema	3.5	133.0%	3.5	121.0%
Iraja/Vista A.	3.0	144.0%	3.0	131.0%
Jacarepagua	3.5	132.0%	3.5	119.0%
J.Botanico	3.5	137.0%	3.5	126.0%
Lagoa	3.5	115.0%	3.5	104.0%
Leblon	3.5	114.0%	3.5	102.0%
Laranjeiras/C.Velho	3.5	129.0%	3.5	117.0%
Madureira	4.0	113.0%	4.0	102.0%
Meier/Lins	3.5	148.0%	3.5	139.0%
Ramos/Leopoldina	3.0	163.0%	3.0	156.0%
S.Teresa	1.0	184.0%	1.0	178.0%
Gloria	3.5	143.0%	3.5	139.0%
S.Cristovão/Benfica	5.5	144.0%	5.5	130.0%
Tijuca	3.5	145.0%	3.5	134.0%
Rio Comprido	2.5	135.0%	2.5	125.0%
Urca	1.0	139.0%	1.0	117.0%

Table 8: Property Taxes After Construction

Locations	Critical Values			
	$\tau d = 6.2\%$		$\tau d = 12\%$	
	Q^*	y^*	Q^*	y^*
Andarai	4.0	280.0%	4.0	375.0%
Grajau	3.0	269.0%	3.0	360.0%
Bangu/C.Grande	3.5	200.0%	3.5	200.0%
Barra/Recreio	3.5	164.0%	3.5	166.0%
Botafogo/Humaita	3.5	172.0%	3.5	175.0%
Cascadura/Piedade	3.0	194.0%	3.0	198.0%
Centro	5.0	177.0%	5.0	178.0%
Copacabana/Leme	3.5	166.0%	3.5	168.0%
Flamengo/Catete	3.5	169.0%	3.5	172.0%
Gavea	3.5	167.0%	3.5	170.0%
Ilha	1.5	170.0%	1.5	174.0%
Ipanema	3.5	164.0%	3.5	167.0%
Iraja/Vista A.	3.0	230.0%	3.0	234.0%
Jacarepagua	3.5	170.0%	3.5	172.6%
J.Botanico	3.5	164.0%	3.5	164.0%
Lagoa	3.5	168.2%	3.5	169.0%
Leblon	3.5	167.7%	3.5	169.0%
Laranjeiras/C.Velho	3.5	164.0%	3.5	166.0%
Madureira	4.0	179.3%	4.0	180.0%
Meier/Lins	3.5	169.0%	3.5	171.0%
Ramos/Leopoldina	3.0	216.0%	3.0	221.0%
S.Teresa	1.0	320.0%	1.0	324.0%
Gloria	3.5	175.0%	3.5	177.0%
S.Cristovão/Benfica	5.5	186.0%	5.5	192.0%
Tijuca	3.5	171.0%	3.5	173.0%
Rio Comprido	2.5	160.0%	2.5	162.0%
Urca	1.0	166.0%	1.0	169.0%

3.5.3 Compatibility of the Theoretical Model with Reality.

As we outlined before in section 3.4, observing the graph of the real business cycle for the real estate market in the city of Rio de Janeiro we can see that January 1996, January 1997, July 1998 and September 1999 are periods of cycle reversion. So, to check if these periods are compatible with the optimal dates for beginning a new building development according to the theoretical model we will next calculate the critical values for each of these periods. Using the individual estimates of the volatility and growth rates for the data beginning respectively in January 1996, January 1997, July 1998 and September 1999 we can calculate the critical values for each zone[20] and critical dates that can be found in the next table. We can note that, for the majority of the zones, the critical values y* are higher during the last two years (July 1998 and September 1999). At first sight, we can interpret this result as a slower construction process in the majority of the zones. However, looking at the evolution of the real estate selling prices in relation to the evolution of the construction costs, as can be seen in figures III-IV[21], we can conclude that the increase in the critical values in the last two years occurred because the real estate selling prices increased more in this period than the construction costs[22].

[20] The upper class zone comprises the following locations: Ipanema, Leblon, Lagoa, Jardim Botânico, Urca, Gávea, Barra and Recreio. The middle class zone comprises the other following locations: Andaraí, Grajaú, Botafogo, Humaitá, Copacabana, Flamengo, Ilha do Governador, Jacarepaguá, Méier, Glória, Laranjeiras, Tijuca and Rio Comprido. Finally, the lower class zone comprises the following locations: Bangu, Campo Grande, Cascadura, Piedade, Centro, Irajá, Vista Alegre, Madureira, Ramos, Leopoldina, So Cristóvão and Benfica. Santa Teresa is analyzed individually because of its unique characteristics, such as the severe legal limitations in this location.

[21] These figures were developed using various construction types according to the location and selling prices, leading us to differential construction costs.

[22] The figures showing the y's evolution of the middle class zone and Santa Teresa and

Zones/ Critical Dates	Upper Class Zone	Middle Class Zone	Lower Class Zone	Santa Teresa
Jan/96	180.07%	180.17%	180.65%	240.0%
Jan/97	200.80%	200.75%	190.98%	260.0%
Jul/98	200.69%	210.40%	210.0%	260.0%
Set/99	220.13%	220.56%	230.6%	270.0%

the analyses of the compatibility of these theoretical results according to the reality of this market can be found in Medeiros (2001).

Finally, we can verify the compatibility of the theoretical results, forecasted by the model, with the reality of the real estate business cycles. To do this we will compare the evolution of the ratio values, y's, (the ratio of the unit cash inflows divided by the unit construction costs) described by figures III-IV with their respective critical values, y^*'s showed in the last table. Whenever we find that $y > y^*$ for a given period we can conclude that it is optimal for the investor to develop at that period. Following this reasoning we can observe that the real estate market is in a high activity period for all the critical dates since $y > y^*$. So, January 1996, January 1997, July 1998 and September 1999 are optimal periods to begin new developments according to the theoretical model. This confirms that the reversion periods of the real business cycles for the real estate market found in graph II are really compatible with the optimal dates for beginning a new building development according to the theoretical model.

So, we can conclude that, in general, the theoretical model is compatible with the reality of the residential real estate market in the city of Rio de Janeiro. We can also conclude that the higher values for the critical values, y^*, found in the last two years (July 1998 and September 1999), are not explained by a slower activity in the construction process, but, on the other hand, it is a result of the increase in the real estate selling prices in relation to the construction costs. So, this market is in a high activity period in the last two years, as can be seen by the evolution of the y's that are, in average, above y^* for this period.

4. Risk Premium of the Unit Cash Inflow.

Finally, we will present the estimates of the implicit risk premium of the net cash inflow described by the parameter λ_i. Setting the value of the risk premium for the construction cost

around 1%, $\lambda_1 = 0.01$, we will calculate implicitly the values for the risk premiums of the property's selling prices for the various zones and critical dates. The table ahead shows the risk premiums for the mainly zones and critical dates that are calculated implicitly using equation 14 and the mean values for all parameters except λ_2. We will also use the real values for the y's of the various zones, as can be seen in the figures III-IV. So, we calculate λ_2 numerically as the argument that minimizes the following function:

$$\lambda_2 = \arg\min \left(y_t^* - y^*\right)^2 \tag{24}$$

Where y^* is the observed value and y_t^* is the solution to equation 14.

Zones/ Critical Dates	Upper Class Zone	Middle Class Zone	Lower Class Zone	Santa Teresa
Jan/96	5.61%	5.6%	8.34%	9.23%
Jan/97	6.35%	6.69%	8.42%	10.41%
Jul/98	8.08%	7.27%	8.7%	10.39%
Set/99	5.88%	3.02%	8.47%	10.45%
Average	6.48%	5.64%	8.48%	10.12%

We can note that the risk premiums required are higher for the lower class zone, especially in Santa Teresa. We can also see that for the majority of the zones the risk premium values are higher for the July 1998 period. The risk premium values for the upper class and middle class zones vary a lot through time while the same values for the lower class zone and Santa Teresa remain relatively stable during the period analyzed. A reasonable explanation for this result is the existence of a wealth effect for the upper class and middle class zones while this effect is smaller or even null for the lower class zone and Santa Teresa.

5. Conclusions.

In this paper, we analyzed how the owner of an undeveloped property chooses the optimal time and optimal density to begin a new property development in order to make an optimal investment. For this purpose, we extended the basic model developed by Williams (1991) in order to include time to build and property taxes before and after construction. We can verify that these extensions have significant effects in the theoretical model.

A detailed empirical analysis of the case of the residential real estate market in the city of Rio de Janeiro was presented. We can confirm that the theoretical results are compatible to the reality of this market, characterized here by the real business cycles. So, we can conclude that the model developed so far is a powerful tool to help investor's decision in the real estate market.

Appendix.

Derivation of the Differential Equation (5)

We saw that the value of the developed property, $P(x_2)$, evolves in response to the stochastic evolution of its net cash inflow, X_2, that is described in the equation (2) at the text. Using Ito's Lemma we obtain the following equation:

$$dP = \left(P' v_2 x_2 + \frac{1}{2} P'' \sigma_2^2 x_2^2 \right) dt + (P' \sigma_2 x_2)\, dz_2 \qquad (A.1)$$

The total return of selling Q units of density is constituted by a capital gain (dP) plus the net cash inflow per unit of time (Qx_2) discounted by a constant value proportional to the developed property (ΘP) because it takes time to build. To preclude riskless arbitrage, this instantaneously riskless portfolio

must yield the riskless rate r as demonstrated in the following equation:

$$\frac{dP + Qx_2 - \Theta P}{P} = r \qquad (A.2)$$

Substituting the expected rate of return for dP described in equation (A.1), in the last equation (A.2), we can conclude that the value of the developed property follows the next differential equation:

$$P' v_2 x_2 + \frac{1}{2} P'' \sigma_2^2 x_2^2 + Q x_2 - (\Theta + r) P = 0 \qquad (A.3)$$

This is the differential equation (5) that appears in the text.

Derivation of the Differential Equation (9)

We saw that the value of the undeveloped property, $V(x)$, is driven by the stochastic evolution of both the unit construction cost, x_1 and the unit cash inflow, x_2 described in the equation (2) at the text. Using Ito's Lemma in a multivariate case, we obtain the following equation:

$$\begin{aligned} dV = \big[& V_1 v_1 x_1 + V_2 v_2 x_2 + \frac{1}{2}(V_{22}\sigma_2^2 x_2^2 + V_{11}\sigma_1^2 x_1^2 \\ & + 2V_{12}\sigma_{12} x_1 x_2)\big] dt + V_1 \sigma_1 x_1 \partial z_1 + V_2 \sigma_2 x_2 \partial z_2 \end{aligned} \qquad (A.4)$$

Because the model takes into account a net cash inflow originated by the use of the undeveloped property, the total return is constituted by a capital gain (dV) plus a net cash inflow per unit of time (βx_2). Again, to preclude riskless arbitrage, the riskless portfolio must yield the riskless rate of return r:

$$\frac{dV + \beta x_2}{V} = r \qquad (A.5)$$

Substituting the expected rate of return for dV described by equation (A.4) in equation (A.5) we can conclude that:

$$V_1 v_1 x_1 + V_2 v_2 x_2 + \frac{1}{2} V_{22} \sigma_2^2 x_2^2 + \frac{1}{2} V_{11} \sigma_1^2 x_1^2 + V_{12} \sigma_{12} x_1 x_2 + \beta x_2 - rV = 0 \qquad (A.6)$$

Derivation of the Differential Equation (16)

We saw that the property tax rate after development, τ_d, can be seen as a discount proportional to the value of the developed property, $(\tau_d P)$. So, to preclude riskless arbitrage, we can conclude that:

$$\frac{dP + Qx_2 - \tau_d P - \Theta P}{P} = r \qquad (A.7)$$

Substituting the expected rate of return for dP, described in equation (A.1), in the last equation (A.7), we can conclude that:

$$P' v_2 x_2 + \frac{1}{2} P'' \sigma_2^2 x_2^2 + Qx_2 - (\Theta + \tau_d + r) P = 0 \qquad (A.8)$$

This is the differential equation (16) that appears in the text.

Derivation of the Differential Equation (19)

As we saw before the property tax rate before development can be seen as a discount proportional to the value of the undeveloped property. So, to preclude riskless arbitrage, we can conclude that:

$$\frac{dV - \tau_a V + \beta x_2}{V} = r \qquad (A.9)$$

Substituting the expected rate of return for dV, described by equation (A.4), in the last equation (A.9), we can conclude that:

$$V_1 v_1 x_1 + V_2 v_2 x_2 + \frac{1}{2} V_{22} \sigma_2^2 x_2^2 + \frac{1}{2} V_{11} \sigma_1^2 x_1^2 + V_{12} \sigma_{12} x_1 x_2 + \beta x_2 - \tau_a V - rV = 0 \qquad (A.10)$$

This is the differential equation (19) that appears in the text.

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