A Real Option Model with Uncertain, Sequential Investment and with Time to Build

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Abstract
This article develops a real option model with uncertain and sequential investment and with time to build. The model includes options to entry and to exit the activity and addresses the maximization problem of a company in view of the investment opportunity. The differential equation of the asset is obtained by using dynamic programming and risk neutral evaluation. Particularly, for the construction period, the differential equation is partial and elliptical, which demands the use of numeric methods. The main results of the article are that (i) with uncertain and sequential investment and with time to build, the waiting value, which creates a gap between the investment decision rule based on NPV and that based on a real option model, may not be very significant and (ii) the increase in uncertainty may anticipate the decision to investment.

Resumo
Este artigo desenvolve um modelo de opções reais com investimento incerto, seqüencial e com tempo de construção. Incorpora-se no modelo as opções reais de investir e abandonar a atividade. O modelo aborda o problema de maximização de uma empresa diante de um investimento com essas características. A equação diferencial do ativo é obtida utilizando programação dinâmica e avaliação neutra ao risco. Em particular, para o período da construção, a equação diferencial é parcial e elíptica, o que demanda a utilização de métodos numéricos. Os principais resultados do artigo são que (i) com investimento incerto, seqüencial e com tempo de construção, o valor de esperar, que gera uma diferença na decisão de investimento baseada no NPV e a baseada em um modelo de opções reais pode não ser significativo e (ii) o aumento da incerteza pode antecipar a decisão de investir.

***Palavras-chave*:** investment under uncertainty; real options; sequential investment; time-to-build; elliptical partial differential equation; numerical methods.

Códigos JEL: D92; G31; G12; G13; E22.

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1. Introduction

This article evaluates a company's investment decision based on a multiple real options model, which combines entry and exit strategies with uncertain and sequential investment and time to build. The company's investment decision takes place in a scenario in which (i) the value of the asset is uncertain, (ii) the total cost of the investment is uncertain, (iii) there is time to build and (iv) the investment is made in stages.

The uncertainty about the value of the asset does not require further explanations. Variations in demand, prices and production costs affect the cash flow generated by assets and, consequently, their values.

Similarly, the cost of the investment to develop a project is also uncertain. As an example, consider that various capital goods in Brazil are quoted in US dollars; therefore, a company in Brazil must take into consideration the foreign exchange risk when it is planning an investment.

The possibility to make investments in stages is also a common characteristic to the many types of investments. Some traditional examples are the launch of new products, research and development of new medicines, exploitation of natural resources, such as the petroleum, the expansion of an industrial plant and real estate developments. The essential characteristic of the sequential investment is the possibility to suspend or abort the investment if the expected value of the complete project decreases or if the cost to complete the investment increases.

Another important characteristic of the model is the inclusion of the time to build, which is present in practically every investment in real assets.

Once the characteristics of the investment are determined, the model addresses the decision of a company that has an option to invest in a project with the characteristics above. Additionally, the company has the options to (i) as the investment is sequential, abort the project during construction period and (ii) when the asset is completed, discontinue the activity.

Economic literature on real options usually addresses the aspects mentioned above. However, most cases take into consideration only one or other aspect and do not consider all the characteristics in a single model. The table below summarizes the hypotheses of some of some articles on real options.

As shown in Figure 1, none of the models analyzes the interaction between the different aspects and real options addressed here. In the model developed in this article, all of the aspects of the table are included.

	Asset Uncertainty	Investment Uncertainty	Investment		Options		
			Sequential	Time to Build	Invest	Exit Construction	Exit Activity
Dixit and Pindyck 1 (1994, Ch. 6)	x				x		x
Dixit and Pindyck 2 (1994, Ch. 6)	x	x			x		
Dixit and Pindyck 3 (1994, Ch. 6)	x				x		x
Dixit and Pindyck 4 (1994, Ch. 10)	x		x	x	x	x	
Dixit and Pindyck 5 (1994, Ch. 10)		x	x	x	x	x	
Medeiros (2001)	x	x		x	x		
Brach and Paxon (2001)	x		x		x	x	
Ilan and Strange (1996)	x			x	x		x
Majd and Pindyck (1987)	x		x	x	x	x	
Milne and Whalley (2000)	x		x	x	x	x	
Ilan and Strange (1998)	x		x	x	x	x	
Kellogg and Charnes (2000)	x		x		x	x	

Figure 1
Summary of the assumptions of the selected articles

The remainder of the article is organized as follows: Section 2 contains some of the stylized facts of the real option literature; Section 3 presents the model, its solution and the results; Section 4 presents the conclusions and; appendix shows some details of the methods used in this paper.

2. Stylized Facts of Real Option Theory

The real option literature is extensive and includes both theoretical and applied articles. Dixit and Pindyck (1994) and Trigeorgis (1996) remain the main textbooks of the real options literature. The most basic models generally consider a company that has an option to invest in a real asset, the return on which is uncertain and considered to follow a stochastic process, but where the investment is immediate, non-sequential and with a known and certain total cost.

In these models, two results have became stylized facts of real options literature: (i) the investment rule based on net present value (NPV) generates mistakes because it does not considers the value of the option to wait for new information; (ii) and the increase in uncertainty postpones the investment.

The traditional NPV investment rule states that an investment should be made if the present value of the future cash flow, discounted at a proper rate, is equal to or higher than zero.

In real option models, the company's decision rules change. In the presence of uncertainty, the company waives the opportunity to wait for new information about the project when it decides to invest. Indeed, the investment decision makes the company loose an asset: the option to invest. This opportunity cost must be measured and included in the investment evaluation.

If we name the value of the option to investment V0, the new investment rule would be represented by (for this result and a review of main concepts in real options see Dixit and Pindyck (1994), Trigeorgis (1996))

$$NPV - V_0 = 0 \tag{1}$$

Accordingly, the investment rule may be presented based on the value of waiting for new information. Therefore, the company only invests when the value of waiting for new information is zero. If we name this waiting value F_0, the rule corresponding to (1) is represented by

$$F_0 = V_0 - NPV = 0 \tag{2}$$

Figure 2 below illustrates the difference between the NPV rule and the rule based on a real options model. The simulation was made taking into consideration a basic real options model in which the uncertainty of the value of the asset is modeled as a Geometric Brownian Motion (GBM) and the investment is certain, immediate and non-sequential.

In the simulation, it is considered a real options model in which the company has the option to invest in an asset that produces a product unit per period, sold at price P_t. To produce this unit, the company incurs in cost C per period, constant in time. The investment necessary to build the asset is known and represented by K. When the company is operational, it has the option to discontinue the activity in exchange for a cash flow E. Should the company discontinue the activity, the company fails to have the option to invest. The uncertainty is included through P_t, which follows a GBM represented by $dP_t = \alpha P_t d_t + \sigma P_t dW_t$, where dW_t is a Wiener process. This stochastic process is transformed into a risk neutral equivalent process, taking into consideration a risk free return rate and a risk premium for asset P represented by ϕ. The values assumed for the parameters in the model are shown in Figure 2. Details and discussions on the model, its parameters and resolution method are addressed in Section 3, in which we develop and solve the model of this article.

According to the traditional NPV rule, the investment would be made with P equaling 1.20. In the real option model, the investment only occurs in $P_H = 2.00$. As the simulation was made with $\sigma = 0.2$, the difference shows that even with median levels of uncertainty, the waiting value and the distance from the NPV rule are significant.

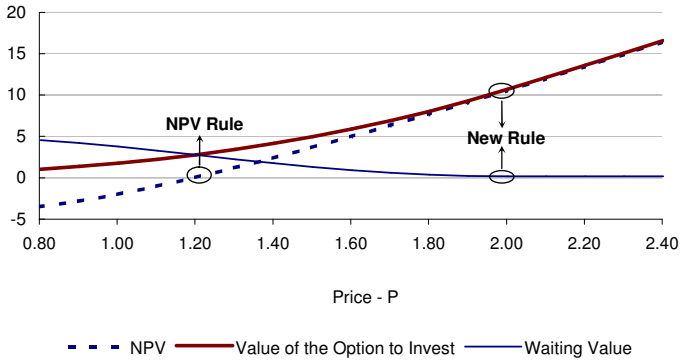

Figure 2
Parameters: $\sigma = 0.2$; $r = 0.04$; $\phi = 0.2$; $\alpha = 0.02$; $K = 4$; $C = 1$; $E = 0$
σ: price volatility; r: risk free rate; ϕ: risk premium; α: drift in price; K: cost of investment; C: unit production cost; E – cash flow if the firm exits activity

The economic intuition of the result is simple. The investment that the company makes is irreversible. With uncertainty, the future return on this investment may not be the expected one. Therefore, postponing the decision to invest to avoid having an asset in a low price scenario has its value. Consequently, the company demands a higher product price, which can provide an extraordinary initial profit to offset the possibility of losses in the future.

Another way of analyzing the effects of uncertainty and of the real option rule is to compare the price that triggers the decision to invest in the model with the classic Marshall's investment criteria (previous use of this concept in the real option literature appeared in Dixit (1992), Dixit and Pindyck (1994)), which affirms that the company must invest when $P \geq \delta K + C$, that is, when the price is higher than the production cost plus the required return on capital. In the real options model, the price level from which the company invests is given by P_H. Figure 3 shows this price for different levels of volatility and the price calculated by the Marshall's criteria.[1]

[1] See Section 3 for discussion about δ.

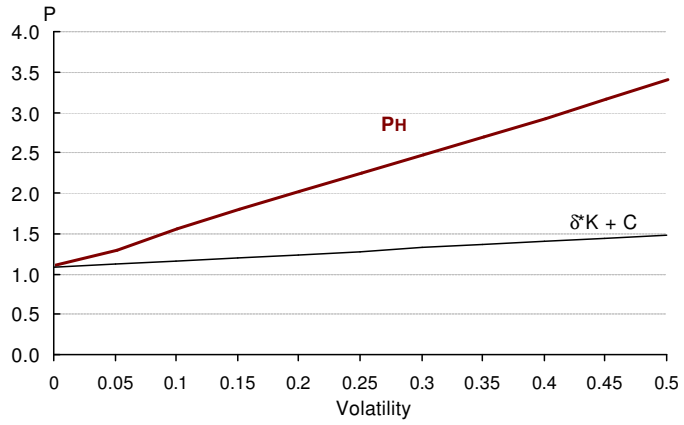

Figure 3
P_H based on the real option model used in figure 2: $r = 0.04$; $\phi = 0.2$; $\alpha = 0.02$; $K = 4$; $C = 1$; $E = 0$.
Marshall's criteria: $K = 4$; $C = 1$; $r = 0.04$;$\phi = 0.2$; $\delta = r + \phi\sigma$

Figure 3 also illustrates another stylized fact of the real option literature: the increase in volatility increases the value of waiting. The difference between P_H and the price of the Marshall's criteria shows such effect: the higher the volatility, the higher the former is in relation to the latter. Note that this fact also implies that the increase in uncertainty may delays the investment. The economic logic is also simple: greater uncertainty makes the option to wait for new information and avoid unfavorable scenarios more valuable and, therefore, postpones the investment.

After the development of the earlier real option models, a series of studies have focused on changing the basic hypothesis assumed in these models, analyzing the effects on the stylized facts pointed above and trying to identify new effects on the investment decisions of the firm.

Most articles mentioned in Figure 1 follow this research line. This article also falls into this category. The remaining sections of the article present the model, solve it and present the results.

3. Sequential and Uncertain Investment and Time to Build

The model developed in this article considers that the investment has three important characteristics: (i) the total investment amount is uncertain; (ii) the construction of the asset demands time to build; (iii) and the decision to invest is made in sequential stages.

In the economic literature, some articles bring together uncertainty of investments with uncertainty of the value of the project. Dixit and Pindyck (1994:6) develops a model in which the investment follows a Geometric Brownian Motion (GBM) and analyzes the company's decision to invest. Medeiros (2001) applies a real option model to the residential real estate market in which the investment cost

also follows a GBM. Brach and Paxson (2001), in a real options model to estimate the value of an R&D project for a new medication, takes into consideration that the total expense on research is known; however they state that this is a too strong simplification from reality.

Brach and Paxson (2001) also discuss the importance of distinguishing the uncertainties of investments between exogenous and technical. The former includes variations in the cost of labor, equipments and other inputs involved in a project; the latter, are the difficulties in the performance of a project that are revealed only as the company invests in the project. For example, the total construction time may take longer than expected and in an R&D program, government regulation and compliance requirements may be higher than those anticipate. The construction period of an asset is known in literature as time to build. The basic characteristic of the time to build is the existence of a lag between the decision to carry out a project and the moment on which this project starts to generate a cash flow. In some industries, the time to build may be very significant, such as the case of pulp and paper, steel mills and hydroelectric plants. The development of new products can also be a long process. Venture capital investments are also a long process, with various stages, for which the moment of the first disbursements occurs much earlier than the moment that the project begins to generate cash. In all these cases, the time to build is important and must be taken into consideration.

Finally, in addition to taking time, investment decisions may be made in many stages and not only in a single occasion. For example, to decide to invest in an R&D program, and invest in an industrial plant to produce the developed product, are different decisions made at different moments. The exercise of an option that assures the right to an R&D program is not complemented by the final asset, but by an option on the construction of the plant.

In particular, venture capital investments are essentially sequential. As they involve new companies, with new products, which still need to be developed and tested on the market, the investments take place in a gradual way and only as the development of the product and market tests are favorable.

Brach and Paxson (2001), Kellogg and Charnes (2000), Milne and Whalley (2000) and Ilan and Strange (1998) develop models in which the investments are made in many stages. However, in none of them there is uncertainty regarding the cost of investment. Dixit and Pindyck (1994:10) includes sequential and uncertain investment in a continuous investment model in the same way as the one adopted here.

3.1 Description of the Model

Take into consideration a company with a specific and perpetual investment opportunity, in a project to produce a product, for which the demand is certain but the price is uncertain.

Fixing the sales at a unit per period, the uncertainty regarding revenues is based on the product's price, which is supposed to follow a Geometric Brownian Motion (GBM):

$$dP_t = \alpha P_t dt + \sigma P_t dW_t \tag{3}$$

At any time, the company may start to invest in the "construction"[2] of the asset. The initial estimated construction cost is represented by K_0. The investment does not take place at once, but sequentially, by way of an expense of I monetary unit, constant in time, per period.

During the construction period, the total investment amount necessary to complete the work is unknown and follows a stochastic process represented by the following equation:[3]

$$dK_t = -Idt + v(IK_t)^{\frac{1}{2}} dZ_t \tag{4}$$

where K_t is the amount of the investment that is necessary for the completion of the construction of the asset, I is the investment per unit of time, v is a constant and dZ_t is a Wiener process that is independent from the dW_t. Note that dZ_t captures the technical uncertainty of the investment because it multiplies the term I. Therefore, realizations of dZ_t only affect K_t if the company invests.

We assume that there is no correlation between dW_t and dZ_t. This seems to be a reasonable assumption: it is unlikely that the firm's specific technical difficulty of completing a project will have much to do with the state of the overall economy and, in particular, with the market price P_t (Dixit and Pindyck, 1994:10).

To include time to build in the model, we assume that the expected initial value of the construction is higher than the investment per period, that is, $K_0 > I$.

Once the decision to start the project is made, the company can, at any time, discontinue the construction in exchange for a known cash flow E_1, which may be positive, in the case that the capital good can be used in other activities, or negative, in the case the company incurs in discontinuation costs, like labor or scrapping costs.

The sequential investment models present in the literature, generally, specify a discrete number of moments in which the company decides whether it continues with the project or not. Here, we assume that the decision takes place continuously. Therefore, one can understand that every dollar invested in the project entitles the company, or gives it the option, to invest another dollar in the next stage of the project.

[2]The "construction" may effectively mean an expense with the construction of an asset, or expenses with R&D, market research, etc.

[3]This equation is a special case of the process $dK = -Idt + g(I, K)dz$. It must attend certain conditions to make economic sense. In particular, the instantaneous variance of dK must be bounded for all finite K (see Dixit and Pindyck 1994:10). This type of process also appears in fixed income models of asset pricing.

When $K_t = 0$, the construction is complete and the company has an asset that produces one unit of a product per period, which is sold at the price P_t. The production cost per unit is given by C, which is constant in time. Should the company decide to discontinue the activity, it will incur in the known cash flow E_2, which, as E_1, may be positive or negative.

Once the company has invested, completed the construction of the asset and then discontinued it, the company is considered to be out of the market and no longer has the opportunity to invest. The same is true if the company aborts the project during the construction phase.

Figure 4 below illustrates the company's alternatives and the cash flow corresponding to each alternative.

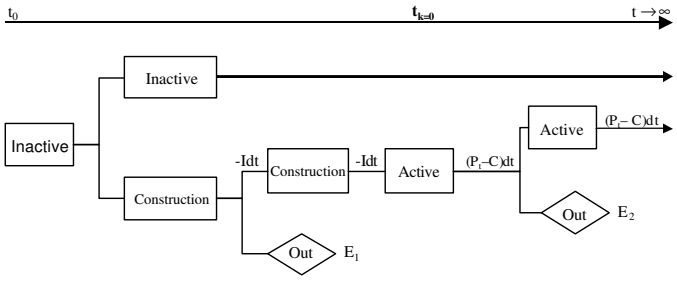

Figure 4
Illustration of the Company's Problem
The problem is continuous and the decisions to invest and abort take place during all t. For illustrative purposes only, a random t was chosen to mark the decision moments

3.2 Solution of the model

The model implies that the company can be at any moment in four possible states: inactive, construction, active or out of the market. The values of the company in the first three phases are named, respectively, V_0, V_1 and V_2. Should it exit activity during construction, the company is worth E_1 and, if it abandon while operational, it is worth E_2.

The first three values are made up of the asset value itself and of the real options that the company owns. The options to defer the investment, abort the project during the investment phase and abort the project when it is operational are included, respectively, in V_0, V_1 and V_2. The values of these options are calculated at a later moment. First, the values V_0, V_1 and V_2 are calculated as functions of the state variables P and K.

In addition to obtaining these values, it is necessary to find the trigger values of P at which it is optimal to exercise the options, that is, P_H, above which it is optimal to invest in the project, P_M, below which it is optimal to abort the project during the construction period and P_L, below which is optimal to abort the operation.

The strategy to solve the model follows the risk neutral evaluation methodology, which is a usual procedure in the financial literature: first, the stochastic processes (3) and (4) are turned into neutral equivalent probability measures processes; second, the decisions to invest are assumed to be made by a company that is risk neutral.

The processes (3) and (4) with risk neutral equivalent probability measure are represented by

$$dP_t = (\alpha - \phi\sigma)P_t dt + P_t dW_t^* \tag{5}$$

$$dK_t = -Idt + v(IK_t)^{1/2} dZ_t \tag{6}$$

where dW_t^* shows that the process is risk neutral and $\phi\sigma$ is the risk premium – excess of average return per unit of standard deviation – associated with dW_t. In the case of dK_t, it is assumed that the technical uncertainty dZ_t has no correlation with the market portfolio, and, therefore, does not have a systemic risk. As a consequence, there is no premium associated with dZ_t and the stochastic process dK_t remains the same. Finally, since dW_t and dZ_t are uncorrelated, dW_t^* remains uncorrelated to dZ_t.

It is worth noting that if P_t is not a tradable security, there is no assurance that its expected rate of return is equal to the market equilibrium rate (see McDolnad and Siegel (1984, 1985), Trigeorgis (1996)). Therefore, σ_p is defined as an implied continuous dividend, or convenience yield, which measures the differences between the expected rate of return from P and its market equilibrium return, which is given by $\mu_p = r + \phi\sigma$. Therefore, $\sigma_p = \mu_p - \alpha$, and we assume that σ_P is positive.

With such interpretation, (5) can be replaced by the most usual way to present risk neutral processes:

$$dP_t = (r - \delta_p)P_t dt + \sigma P_t dW_t^* \tag{7}$$

Once the stochastic processes are defined in a risk neutral equivalent measure, the values V_0, V_1 and V_2 are given by the optimal decisions of a risk neutral company.

Inactive: $V_0(P_t, K_t)$

The value of the company while it is inactive, V_0, is defined by the maximization problem of a risk neutral company that chooses between remaining inactive or starting the construction of the asset:

$$V_0(P_t, K_t) = \max \mathbb{E}_0 \left\{ e^{-rdt} V_0(P_t + dP_t, K_t + dK_t); V_1(P_t, K_t) \right\} \tag{8}$$

The first term between the brackets reflects the decision to remain inactive; as there is no immediate flow associated with this decision, the return is the expected

variation in the company's value. The second term is the value of the company in the construction phase.

For the price interval $P_t < P_H$, the company remains inactive and

$$V_0(P_t, K_t) = \mathbb{E}_0 \left\{ e^{-rd_t} V_0(P_t + dP_t, K_t + dK_t) \right\} \tag{9}$$

Expanding the expression between brackets by applying the Ito's Lemma, it follows that

$$V_0 = \mathbb{E}_0 \Bigg\{ (1 - rdt) \Big[V_0 \frac{\partial V_0}{\partial P} dP + \frac{1}{2} \frac{\partial^2 V_0}{(\partial P)^2} (dP)^2 + \frac{\partial V_0}{\partial K} dK + \frac{1}{2} \frac{\partial^2 V_0}{(\partial K)^2} (dK)^2 \Big] \Bigg\} \tag{10}$$

Replacing equations (5) and (6) in (10), with $I = 0$, since the company is inactive, and applying the expectation operator

$$V_0 = (1 - rdt) \left\{ V_0 + (\alpha - \phi\sigma) P \frac{\partial V_0}{\partial P} dt + \frac{1}{2} \sigma^2 P^2 \frac{\partial^2 V_0}{(\partial P)^2} dt \right\}$$

$$0 = \frac{1}{2} \sigma^2 P^2 \frac{\partial^2 V_0}{(\sigma P)^2} + (\alpha - \phi\sigma) P \frac{\partial V_0}{\partial P} - r V_0 \tag{11}$$

(11) is the differential equation that $V_0(P, K)$ must follow while $P < P_H$. The equation is subject to the following boundary conditions

$$V_0(0, K) = 0 \tag{12}$$

$$V_0(P_H, K) = V_1(P_H, K) \tag{13}$$

$$\frac{\partial V_0(P_H, K)}{\partial P} = \frac{\partial V_1(P_H, K}{\partial P} \tag{14}$$

The condition (12) states that when P equals 0, the value of the opportunity to invest, V_0, is zero. The conditions (13) and (14) are, respectively, the value matching and smooth pasting conditions suitable for the problem. These conditions appear in all real options models. Both can be justified by arbitrage arguments regarding an optimal stopping problem. Value matching means that the value of the unknown function V_0 matches the value of the known termination payoff function V_1. Smooth pasting, or "high-order contact", condition means that not only the values but also the derivatives of the functions V_1 and V_2 must be equal at the boundary. According to Dixit and Pindyck (1994, pp. 109): "While continuity is

very intuitive, continuity of slopes or smooth pasting is more subtle and remarkable. However, the argument for it is somewhat technical". Appendix 3 presents a non-rigorous demonstration of both conditions.

The equation (11) is an ordinary differential equation whose general solution is given by

$$V_0(P,K) = A_1 P^{\beta_1} + A_2 P^{\beta^2} \tag{15}$$

where A_1 and A_2 are unknowns, and β_1 and β_2 are the roots of the quadratic form

$$1/2\sigma^2\beta(\beta-1) + (\alpha - \phi\sigma)\beta - r = 0 \tag{16}$$

The solutions of this quadratic form are given by

$$\beta = 1/2 - (\alpha - \phi\sigma)/\sigma^2 \pm \sqrt{((\alpha - \phi\sigma)/\sigma^2 - 1/2)^2 + 2r/\sigma^2} \tag{17}$$

Consider (16) as a function of β, named $\varphi(\beta)$. As this function is of second degree, it has two roots, represented by (17); in addition, as $\varphi^{''}(\beta) = \sigma^2/2 > 0$, $\varphi(\beta)$ is a convex function of β. Note that $\varphi(0) = -r < 0$ and $\varphi(1) = (\alpha - \phi\sigma) - r = -\delta < 0$ (in which $\delta = \mu - \alpha > 0$). Therefore, $\varphi(\beta)$ has one negative root and one positive root higher than 1. Name β_1 the positive root and β_2 the negative one: $\beta_2 < 0 < 1 < \beta_1$.

Since $\beta_2 < 0$, (12) implies that $A_2 = 0$. Then $V_0(P,K)$ is given by

$$V_0(P,K) = A_1 P^{\beta_1} \tag{18}$$

At a first moment, the function (18) surprises, as it does not have K as an argument. But the constant A_1, determined by the value matching and smooth pasting conditions, depends on K. In addition, P_H also depends on K, as it will be seen in the complete resolution of the model. Intuitively, one can expect the deferment of the investment with low P or high K. In the case of K, a higher K implies that the expected construction period will be longer and the funds paid out higher. These require a higher P_H to make the profitability of the investment more likely and hence incentive the company to initiate the construction period.

Construction: $V_1(P,K)$

The solution of V_1 is given by the maximization problem of a risk neutral company, which chooses between continuing the construction of the asset or suspend it:

$$V_1(P,K) = \max \mathbb{E}_0 \left\{ -Idt + e^{-rdt} V_1(P + dP, K + dK); E_1 \right\} \tag{19}$$

The first term between the brackets reflects the decision to continue the construction. The company's return is represented by the immediate negative cash flow to continue the investment, $-Idt$, and the instantaneous appreciation of the

company's value. The second term is the value of the company should it abort the project.

For the price interval $P > P_M$, the company continues to invest. In this interval,

$$V_1(P,K) = \mathbb{E}_0\left\{-Idt + e^{-rdt}V_1(P+dP, K+dK)\right\} \tag{20}$$

As in the previous section, expanding the expression between brackets by way of the use of Ito's Lemma gives,

$$\begin{aligned} V_1 &= E_0\Bigg\{-Idt + (1-rdt)\Big[v_1\frac{\partial V_1}{\partial P}dP + \frac{1}{2}\frac{\partial^2 V_1}{(\partial P)^2}(dP)^2 \\ &+ \frac{\partial V_1}{\partial K}dK + \frac{1}{2}\frac{\partial^2 V_1}{(\partial K)^2}(dK)^2\Big]\Bigg\} \end{aligned} \tag{21}$$

Replacing the equations (5) and (6) in equation (21), with the difference that now $I \neq 0$, and applying the expectation operator:

$$\begin{aligned} V_1 = &- Idt + (1-rdt)\Bigg\{V_1 + \Big[(\alpha-\phi\sigma)P\frac{\partial V_1}{\partial P} + \frac{1}{2}\sigma^2P^2\frac{\partial^2 V_1}{(\partial P)^2} \\ &- I\frac{\partial V_1}{\partial K} + \frac{1}{2}v^2IK\frac{\partial^2 V_1}{(\partial K)^2}\Big]dt\Bigg\} \end{aligned}$$

$$0 = \frac{1}{2}\partial^2P^2\frac{\partial^2 V_1}{(\partial P)^2} + \frac{1}{2}v^2IK\frac{\partial^2 V_1}{(\partial K)^2} + (\alpha-\phi\sigma)P\frac{\partial V_1}{\partial P} - I\frac{\partial V_1}{\partial K} - rV_1 - I \tag{22}$$

V_1 is the solution of the partial differential equation (22), which is subject to

$$V_1(P,0) = V_2(P) \tag{23}$$

$$P \to \infty \Rightarrow V_1(P,K) = V_2(P)e^{-rK/I} - \int_0^{K/I} Ie^{-rt}dt \tag{24}$$

$$V_1(PM,K) = E_1 \tag{25}$$

$$\frac{\partial V_1(P_M,K)}{\partial P} = 0 \tag{26}$$

$$\frac{\partial V_1(P_M,K)}{\partial K} = 0 \tag{27}$$

The condition (23) states that when the construction is completed $(K = 0)$, the company has the same value as the active company. Condition (24) establishes that, with a high P, the construction is unlikely to be discontinued and, consequently, the value of the company during construction is the present value of the active company less the amount that remains to be invested.

Conditions (25), (26) and (27) are the value matching and smooth pasting conditions. It is also worth noting that, as P_H, the trigger price P_M depends on K.

Finally, (22) is a partial differential equation with free boundary, more precisely an elliptical equation $(0 < 1/4\sigma^2 P^2 v^2 IK)$, which demands the use of numerical methods for its solution.

Active: $V_2(P)$

The value of the company while active is defined by the maximization problem of a risk neutral company that chooses between continuing its operation and exiting the activity. Note that as the active company does not need to make any more investment its value depends on P only:

$$V_2(P) = \max \mathbb{E}_0 \left\{ (P - C)dt + e^{-rdt} V_2(P + dP); E2 \right\} \tag{28}$$

In the equation above, the first term between the brackets reflects the decision to continue operation, which generates an immediate flow, $P - C$, and the instantaneous variation of the company's value. The second term is the value of the company should it abort the operation.

The price P_L, which is part of the model's solution, is the trigger below which the company would rather prefer to withdraw from the activity. Therefore, in the price interval $P_L < P$, the company remains active and its value is given by the first term of the equation (28):

$$V_2(P) = \mathbb{E}_0 \left\{ (P - C)dt + e^{-rdt} V_2(P + dP) \right\} \tag{29}$$

Expanding the expression $V_2(P_t + dPt)$ by way of the use of Ito's Lemma,

$$V_2 = E_0 \left\{ (P - C)dt + (1 - rdt) \left[V_2 + \frac{\partial V_2}{\partial P} dP + \frac{1}{2} \frac{\partial^2 V_2}{(\partial P)^2} \right] \right\} \tag{30}$$

Replacing equation (5) in equation (30) and applying the expectation operator, then

$$V_2 = (P - C)dt + (1 - rdt) \Big[V_2 + (\alpha - \phi\sigma) P \frac{\partial V_2}{\partial P} dt + \sigma^2 P^2 \frac{1}{2} \frac{\partial^2 V_2}{(\partial P)^2} dt \Big]$$

$$\sigma^2 P^2 \frac{1}{2} \frac{\partial^2 V_2}{(\partial P)^2} + (\alpha - \phi\sigma) P \frac{\partial V_2}{\partial P} - rV_2 + P - C = 0 \tag{31}$$

Equation (28) is subject to

$$P \to \infty \Rightarrow V_2(P) \to X \tag{32}$$

$$V_2(PL) = E_2 \tag{33}$$

$$\frac{\partial V_2(P_L)}{\partial P} = 0 \tag{34}$$

Condition (32) states that, the higher the P, the lower the probability that the company will discontinue the activity. In this case, the value of the option to exit the activity tends to zero and the company's value tends to the intrinsic value of the asset $(X = P/\delta - C/r)$. Conditions (33) and (34) are the value matching and smooth pasting conditions proper for the boundary P_L where the company is indifferent regarding remaining active and discontinuing the activity.

Equation (31) is a non-homogeneous ordinary differential equation. The solution of the homogeneous part has the same form as (15) but with different unknowns.

The particular solution is represented by $P/\sigma - C/r$, which has an immediate economic interpretation: it is the value of the company, assuming that it will never exercise the exit option. To see that note that $P/\sigma - C/r$ is the solution for $\int_0^\infty ((Pe^{\alpha-\phi\sigma} - C)e^{-rt})dt$. Therefore, the general solution of (31) is given by

$$V_2(P) = B_1 P^{\beta_1} + B_2 P^{\beta_2} + P/\sigma - C/r \tag{35}$$

in which B_1 and B_2 are unknowns to be determined and, as in the equation (15), $\beta_2 < 0 < 1 < \beta_1$ are given by (17).

As $\beta_1 > 0$, (32) implies that $\beta_1 = 0$. Then

$$V_2(P) = B_2 P^{\beta_2} + P/\sigma - C/r \tag{36}$$

At this point, it is interesting to identify that the company's value is divided into the intrinsic value of the asset, given by the present value of its future cash flow, X, and into the value of the exit option, denoted by F_2.

$$X = P/\sigma - C/r \tag{37}$$

$$F_2(P) = B_2 P^{\beta_2} \tag{38}$$

Replacing (36) in (33) and (34), we obtain a differential equation system. Its resolution provides the expressions for the unknown factors B_2 and P_L.

$$B_2 P_L^{\beta_2} + PL/\sigma - C/r = E_2 \tag{39}$$

$$\beta_2 B_2 P_L^{\beta_2 - 1} + 1/\sigma = 0 \tag{40}$$

From (39) and (40) result that

$$PL = \beta_2/(\beta_2 - 1)\sigma/r(C + E_2 r) \tag{41}$$

$$B_2 P^{\beta_2} = (P/PL)^{\beta^2}(C + E_2 r)/((1 - \beta_2)r) \tag{42}$$

3.3 Numeric solution

The complete solution of the model is obtained through backward resolution. First, we solve the problem of the active company, then that of the company in the construction period and, finally, the problem of the company while inactive.

Equations (42), (41) and (36) are the analytical solution of the active company's problem. The solution for the problem during the construction period demands the numerical solution of the differential equation (22). Finally, (18) is the analytical solution of the problem of the company while it is inactive and conditions (13) and (14) determine A_1 e P_H. The Appendix describes in detail the solutions of the equations (22) and (18).

The first question to be answered regards the effect of the model's hypothesis on the two stylized fact of the real options theory presented in Section 2: (i) real options change the investment decision rule and (ii) the higher the uncertainty, the higher the distance between the NPV rule and the real option rule.

The model developed in this article suggests that these results are partially reversed. We annunciate the first result below:

Result I: With uncertain and sequential investment and time to build, the value of waiting for new information decreases and, consequently, the investment decision moves closer to the NPV rule.

As previously discussed, the gap between the investment decision in a real options model and the NPV rule can be measured by the distance between the price triggers the investment, P_H, and the price given by Marshall's criteria. As Figure 5 below shows, P_H computed using the model of this section, named P_H - Section 3, remains above the price of the Marshall's criteria for most of the volatility interval. However the distant between both prices is considerably reduced when compared to the case of a standard real option model, as the one presented in section 2 (the curve labeled P_H - Section 2 in Figure 5).

The second result concerns the effect of volatility and we annunciate it below.

Result II: For high levels of volatility, the uncertain and sequential investment and time to build implies that an increase in volatility reduces the price that triggers the investment.

Again, as Figure 5 illustrates, for the volatility interval higher than 15%, an increase in volatility implies a lower P_H, which is a complete inverse sign to that obtained in the basic real options model. In other words, an increase in uncertainty can lead the company to invest earlier.

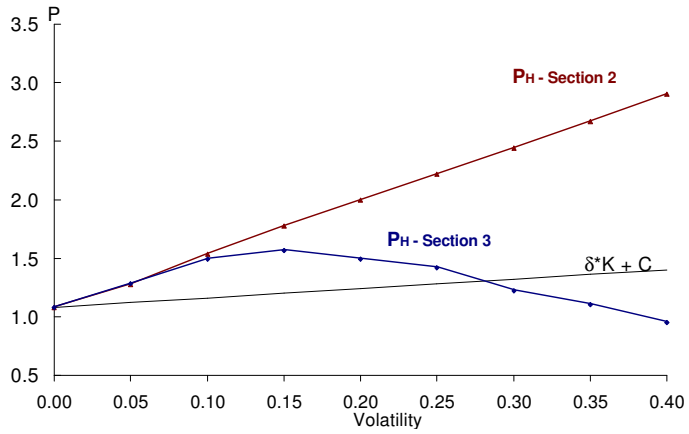

Figure 5
Parameters: $r = 0.04; = 0.2; = 0.02; C = 1; = 0.2; K = 4; I = 1; E1 = E2 = 0$

In the basic real options model, the incentive to defer the investment arises because the uncertainty has an asymmetrical effect on the benefits and costs of postponing the investment. On one hand, upon waiting, the company avoids adverse future scenarios. As higher uncertainty increases the probability of lower prices scenarios, the advantage of waiting grows with uncertainty. On the other hand, the opportunity cost is the non-appropriation of the instantaneous cash flow generated by the asset. In the absence of time to build, if the price goes up, the company immediately enters the market and operates in the favorable scenario. Therefore, the opportunity cost of waiting is independent from uncertainty.

Time to build changes these relations. Now, as the company cannot take immediate advantage of favorable scenarios, the opportunity cost of waiting also depends on uncertainty because if the company does not invest, it bears the risk of being out of the market in the case of a price increase in the future. Therefore, the opportunity cost to defer the investment depends on the future price of the product. In addition, a higher uncertainty increases the chances of favorable extreme events occurring. Consequently, the uncertainty increases the opportunity cost to defer the investment. Additionally, the possibility to suspend the construction limits the

effect of the unfavorable scenarios and decreases the benefit to defer the investment compared to the case that there is no exit option. If the benefit and the cost to defer are positively affected by volatility, the final effect becomes ambiguous. From Figure 5, we note that the second effect can prevail and an increase in uncertainty leads the company to anticipate the investment. This is the main economic intuition of Results I and II presented above.

To illustrate the robustness of these results, we simulated the model for different parameters assumptions. Figure 6 presents the outcome of these simulations. In particular, we are interested in different values and relations for the drift of the process of the price (α), the risk premium (ϕ), and the implied convenience yield ($\delta_p = \mu_p - \alpha = r + \phi\sigma - \alpha$). The first two columns assumes different values for α and ϕ. Note that this implies that δ_p and the risk neutral drift ($\alpha - \phi\sigma$) in the equation (5) change as the volatility changes. In the last three columns we keep δ_p and $(\alpha - \phi\sigma)$ constants by changing α. All simulations support the Results I and II commented above.

	PH Section 3				
	ϕ=0.1	ϕ=0.1	δ = 0.06	δ = 0.04	δ = 0.02
σ	α=0.02	α=0.03	α–ϕσ=–0.02	α–ϕσ=0	α–ϕσ=0.02
0.05	1.28	1.28	1.42	1.35	1.28
0.10	1.49	1.49	1.49	1.49	1.49
0.15	1.57	1.65	1.49	1.57	1.65
0.20	1.57	1.65	1.49	1.57	1.73
0.25	1.49	1.49	1.42	1.49	1.57
0.30	1.28	1.35	1.28	1.35	1.35
0.35	1.11	1.16	1.11	1.16	1.16
0.40	0.95	0.95	0.95	1.00	1.00

* Parameters that are not mentioned have the same value as in Figure 5

Figure 6

Part of these results appears in the economic literature. Ilan and Strange (1996) presents a similar result to that described above. The authors develop a model with time to build and with an option to suspend construction. Their results also show that an increase in uncertainty can reduce P_H. The explanation of the authors is similar to the arguments of the previous paragraph.

However, for high volatility values, the results of Ilan and Strange (1996) follow the traditional results of the real options theory: the increase in uncertainty increases the trigger price P_H. The same does not happen here. As Figures 5 and 6 illustrate, even for high volatility (for example, above 0.30), an increase in uncertainty continues to encourage the investment through the decrease in the threshold price P_H.

The difference between both models arises from the inclusion of the sequential investment. The possibility to make investment in sequential stages further re-

duces the benefit to differ the investment because it dilutes in time the sunk cost of the investment. If we also add the possibility to suspend the construction, the irreversibility of the investment, which is essential to the real options theory, decreases and, therefore, the incentive to wait is reduced.

Figure 7 illustrates this effect. In all scenarios, the total investment is initially estimated in 4 monetary units, and the initial expected time to build always remains at 4 years. However, in each scenario, a different value is assumed for the initial investment, named I_0, which takes place at the moment on which the company decides to invest. The higher this initial value, the less sequential is the investment.

Assumptions Regarding the Investment				**P_H - different levels of price volatitility**			
Total K	Initial Investment	Remaining K	I	σ = 0.1	σ = 0.2	σ = 0.3	σ = 0.4
4	0.0	4.0	1.00	1.492	**1.492**	1.221	0.951
4	1.0	3.0	0.75	1.492	**1.492**	1.284	1.000
4	2.0	2.0	0.50	1.492	**1.568**	1.350	1.051
4	3.0	1.0	0.25	1.492	**1.649**	1.419	1.221
4	3.5	0.5	0.13	1.492	**1.733**	1.733	1.568

Figure 7
Parameters: $r = 0.04; = 0.2; = 0.02; K/I = 4; C = 1; = 0.2; E1 = E2 = 0$

The simulation shows that, the lower the initial investment, the lower is the trigger price P_H. In the base case ($\sigma = 0.2$), if the investment is totally sequential ($I_0 = 0$), P_H is equal to 1.492 and if less sequential (I_0=3.5, which represents 87% of the total investment), P_H = 1.733: a 16% differential, which indicates that if the investment is not sequential the company demands a higher product price to begin the construction.

Another important fact is that the volatility increases this result: for $\sigma = 0.1$, there is no price difference; however, for $\sigma = 0.4$, the price difference between the most and the less sequential cases is equal to 65%.

This result shows that, in a real option model, the sequential investment reinforces the effect of the time to build on the investment decisions. This aspect has not appeared before in the literature and is the main result of this paper. Below, we formally enunciated it.

Result III: Time to build generates situations in which the increase in uncertainty may anticipate the decision to investment. The inclusion of sequential investment reinforces this result.

At this point, it is important to compare the results obtained here with those shown by Milne and Whalley (2000), Majd and Pindyck (1987) and Dixit and Pindyck (1994). All these papers assume sequential investment, time to build and the option to suspend the construction. Yet the results are different from those

obtained in this article. Majd and Pindyck (1987) and Dixit and Pindyck (1994) show that these hypotheses favors the stylized results of the real options theory, that is, the difference between the real option rule and the NPV rule is greater and an increase in uncertainty increases the incentive to wait. Milne and Whalley (2000) reaches results that are similar to ours regarding the NPV rule but they find a positive correlation between uncertainty and the trigger price P_H.

The modeling of the uncertainty of the asset is the main difference between these articles and our work. Whereas here, the uncertainty occurs at the selling price P, in those articles, the asset V itself follows a stochastic process. Accordingly, in those papers the cost to defer the investment is given by the implied convenience yield δ of holding asset V, which they assume to be constant. Therefore, the increase in volatility does not affect the waiting cost and all the effects discussed above do not take place.

Finally, regarding the uncertainty of the total cost of investments, given by, Figure 8 below indicates that the effect on the firm's decision is not relevant. The price threshold that triggers the investment, P_H, for the base model ($v = 0.2$) discussed in this section is presented in the first column for different scenarios of price volatility. The second column shows the P_H for a simulation in which $v = 0$, that is with no investment cost uncertainty. As can be noted in the figure, there are no significant differences between the threshold prices P_H presented in each column, what indicates that the uncertainty of the total cost of investments has little importance for the firm investment decision in the model.

σ	**PH**	
	vega = 0.2	vega = 0
0.05	1.28	1.28
0.10	1.49	1.49
0.15	1.57	1.49
0.20	1.49	1.49
0.25	1.42	1.42
0.30	1.22	1.28
0.35	1.11	1.16
0.40	0.95	0.95

Figure 8
Parameters: $r = 0.04; = 0.2; = 0.02; K = 4; I = 1; C = 1; E1 = E2 = 0$

The way we modeled the uncertainty of the investment generates two effects: (i) greater uncertainty regarding the total cost of the construction decreases the investment and (ii) as we only learn about the cost while investing, there is a kind of learning by doing experience that motivates the beginning of the construction.

In the simulations made, these two effects seem to cancel each other or, should one of the two effects prevail, it has a reduced impact when compared to the effects of the time to build and the possibility of making the investment in stages. This result is another contribution of the article.

Result IV: the volatility of the investment cost, in a model that only has technical uncertainty and with time to build and sequential investment, does not have a material impact on the company's investment decision.

3.4 Applications

The results of the real options models can be used, among other applications, to interpret economic phenomena involving investment decisions.

Ilan and Strange (1996) associates the results obtained with industries in which the time to build is long and there is a chronic excess of capacity. Focusing on companies with long-term investments horizon, like those from the commercial offices construction, electricity generation, pulp and paper and steel sectors, they try to associate their models with situations in which there is a chronic excess of capacity.

The model of this article also suggests an explanation for another economic phenomenon. In the second half of the '90s, companies from sectors related to information technology, Internet and telecommunications went through two important events: (i) the value of their shares were highly appreciated and (ii) these sectors attracted large volumes of investments, which, in some cases, were supported by weak NPV analysis.

Some authors used the option theory to explain the high value of the shares of these companies (see, as an example, Schwartz and Moon (2000)). The results obtained here suggest that a real option model can also provide an explanation for the phenomenon of making an investment even in view of unattractive expected cash flows.

The fast evolution of new technologies such as the Internet and mobile telecommunication generated great uncertainty regarding the future return on the enterprise ventures in these sectors. Additionally, investors proved to be less demanding regarding the immediate profitability of their investments, which may be interpreted as an increase in time to build. One can also argue that the investments, for example, made on many start-ups enterprises and the purchase of concessions to exploit telecommunication services, were initial investment expenses of a longer investment program, which would occur in many stages.

As the results of the previous section showed, the combination of longer time to build, sequential investment and increase in uncertainty seen at the end of the '90s could have accelerated investments. In other words, the uncertainty regarding the return on new technologies encouraged investors to make the investment in projects that were hardly justified by an NPV analysis.

4. Conclusion

In a traditional real option model, the investment is known, immediate and made just once. Economics literature has been addressing the effects of changing these hypotheses. Most articles explore only one of these alternatives. This article contributes to the literature by exploring all these hypotheses together, that is, we present a model in which the investment is at the same time uncertain, sequential and needs construction time.

With the use of dynamic programming and a risk neutral approach, the differential equations for the company's value in the inactive, construction and active stages were obtained. The differential equation of the problem of the company while in construction is partial and elliptical, which requires the use of numeric methods.

The main results of the numeric simulation were:

- Result I: with uncertain and sequential investment and time to build, the value of waiting for new information decreases and, consequently, the decision to invest approximates the NPV rule;
- Result II: an increase in uncertainty may anticipate the decision to investment;
- Result III: the sequential investment strengthens the effect that time to build has on investment decision;
- Result IV: given the other hypotheses of the model, the technical uncertainty of the investment cost does not affect significantly the company's investment decision.

Finally, it is pointed out that among these four results, two stand out as this article's main contributions for the literature: (i) the sequential investment strengthens the effect of time to build, especially when volatility is high and; (ii) in view of other hypotheses, the technical uncertainty of the investment has little influence on the company's decision to invest.

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Appendix 1

This appendix presents the numerical solution of the partial differential equation (22) transcribed below together with the boundary conditions, through the method of finite differences named projected successive over-relaxation (Psor) (for the numerical solutions see Thomas (1995), Morton and Mayers (2005), Wilmott et al. (1995), Wilmott (1998)).

$$0 = \frac{1}{2}v^2 IK\frac{\partial^2 V_1}{(\partial K)^2} + \frac{1}{2}\sigma^2 P^2\frac{\partial^2 V_1}{(\sigma P)^2} - I\frac{\partial V_1}{\partial K} + (\alpha - \phi\sigma)P\frac{\partial V_1}{\partial P}rV_1 - I \quad \text{(A.1)}$$

$$V_1(P,0) = V_2(P) \quad \text{(A.2)}$$

$$P \to \infty \Rightarrow V_1(P,K) = V_2(P)e^{-rK/I} - \int_0^{K/I} Ie^{-rt}dt \quad \text{(A.3)}$$

$$V_1(P_M,K) = E_1 \quad \text{(A.4)}$$

$$\frac{\partial V_1(P_M,K)}{\partial P} = 0 \quad \text{(A.5)}$$

$$\frac{\partial V_1(P_M,K)}{\partial K} = 0 \quad \text{(A.6)}$$

Equation (A.1) is an elliptic partial equation with free boundary. In finance, the most frequent differential equations are parabolic. In the real option literature, differential equations are also usually parabolic and when they are elliptical, a simple transformation into ordinary equations is possible.

The main difference with the numerical resolution of an elliptical PDE is that once the grid for the variables is defined, all the internal points must be solved at once; this is not the case with parabolic ones, where the points are solved one step in time at a time. The problem also requires that four boundary conditions be provided, and not only two and one for the starting value, as in the case of the parabolic equations.

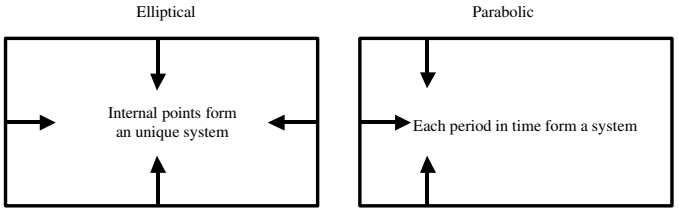

Figure A.1
Illustration of elliptical and parabolic PDEs numerical solution

The finite differences method consists in approximating the continuous problem (A.1) – (A.6) through a discrete solution. First, we define the following variables: $y = lnP$; $x = 2\sqrt{K}$ e $F(y,x) \equiv V_1(P,K)$. Thus,

$$\frac{\partial V_1}{\partial P} = \frac{\partial F}{\partial y}\frac{1}{P} \quad ; \quad \frac{\partial^2 V_1}{\partial P^2} = \frac{\partial^2 F}{\partial y^2}\frac{1}{P^2} - \frac{\partial F}{\partial y}\frac{1}{P^2} \tag{A.7}$$

$$\frac{\partial V_1}{\partial K} = \frac{\partial F}{\partial x}\frac{1}{\sqrt{K}} \quad ; \quad \frac{\partial^2 V_1}{\partial K^2} = \frac{\partial^2 F}{\partial x^2}\frac{1}{K} - \frac{1}{2}\frac{\partial F}{\partial x}\frac{1}{K\sqrt{K}} \tag{A.8}$$

Replacing the new variables and the equations (A.7) and (A.8) in the problem (A.1) – (A.6), and defining that, $d(x) \equiv (-1 - \frac{v^2}{4})\frac{2I}{x}$ and $e \equiv (\alpha - \phi\sigma - \frac{1}{2}\sigma^2)$:

$$\frac{1}{2}v^2 I\frac{\partial^2 F}{\partial x^2} + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2} + d(x)\frac{\partial F}{\partial x} + e\frac{\partial F}{\partial y} - rF - I = 0 \tag{A.9}$$

$$F(y,0) = V_2(e^y) \tag{A.10}$$

$$y \to \infty \Rightarrow F(y,x) \to V_2(e^y)e^{-r\frac{K}{I}} - \int_0^{\frac{K}{I}} Ie^{-rt}dt \tag{A.11}$$

$$IF(y^*,x) = E_1 \tag{A.12}$$

$$\frac{\partial F(Y^*,X)}{\partial Y} = 0 \tag{A.13}$$

$$\frac{\partial F(y^*,x)}{\partial x} = 0 \tag{A.14}$$

where y^* represents the free boundary.

The next step is to construct a discrete variable space and introduce a value grid in the domain of variables y and x. As a matter of convenience, a uniform grid is used in which $\Delta y = y_{\max}/m$ and $\Delta x = x_{\max}/n$, as shown in Figure A.2 below.

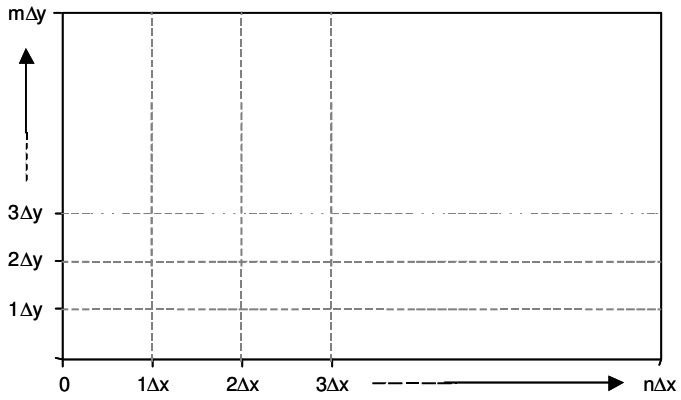

Figure A.2

Next, the problem (A.9) – (A.14) is approximated in the above grid. To that end, the central differences are used both for the second-order derivatives and for the first-order ones.

$$\partial^2 F/\partial y^2 = (U_{i+1,j} - 2_{Ui,j} + U_{i-1,j})/\Delta y^2 \tag{A.15}$$

$$\partial^2 F/\partial x^2 = (U_{i,j+1} - 2_{Ui,j} + U_{i,j-1})/\Delta x^2 \tag{A.16}$$

$$\partial F/\partial y = (U_{i+1,J} - U_{i-1,J})/2\Delta y \tag{A.17}$$

$$\partial F/\partial x = (U_{i,j+1} - U_{i,j-1})/2\Delta x \tag{A.18}$$

By replacing (A.15) – (A.18) in (A.9) – (A.14) and defining $s = \Delta y/\Delta x$

$$U_i^j = \frac{1}{\beta_0}(\Delta y^2 I - \beta_1 U_i^{j+1} - \beta_2 U_i^{j-1} - \beta_3 U_{i+1}^j - \beta_4 U_{i-1}^j) \tag{A.19}$$

where

$$\beta_0 = -(\sigma^2 + s^2 v^2 I + r\Delta y^2) \tag{A.20}$$

$$\beta_1 = s^2\Big\{\frac{1}{2}v^2 I + \frac{d(x)}{2}\Delta x\Big\} \tag{A.21}$$

$$\beta_2 = s^2\Big\{\frac{1}{2}v^2 I - \frac{d(x)}{2}\Delta x\Big\} \tag{A.22}$$

$$\beta_3 = \frac{1}{2}\sigma^2 + \frac{e}{2}\Delta y \tag{A.23}$$

$$\beta_4 = \frac{1}{2}\sigma^2 - \frac{e}{2}\Delta y \tag{A.24}$$

The boundary conditions become:

$$U_i^0 = V_2(e^{i\Delta y}) \tag{A.25}$$

$$U_m^j = V_2(e^{m\Delta y})e^{-r\frac{(j\Delta x)\hat{}2/4)}{I}} - \int_0^{\frac{(j\Delta x)\hat{}2/4)}{I}} Ie^{-rt}dt \tag{A.26}$$

$$U_{i^*}^j = E_1 \tag{A.27}$$

$$\frac{U_{i^*}^j - U_{i^*-1}^j}{\Delta y} = 0 \tag{A.28}$$

$$\frac{U_{i^*}^{j+1} - U_{i^*}^j}{\Delta x} = 0 \tag{A.29}$$

Note that equations (A.27) – (A.29) establish the free boundary and can be solved by following the methodology adopted in financial literature to solve American financial options, that is, using the *Projected* SOR (see Wilmott et al. (1995); Wilmott (1998)).

The solution of problem begins by the adoption starting values for all of the points U_i^j inside the value grid and by calculating the values of U_i^j for $j = 0(\forall i)$ and for $i = m(\forall j)$ through (A.25) and (A.26) respectively, that is, the boundary conditions for $V_1(P, 0)$ and $V_1(P, K)$ with $P \to \infty$ are calculated. Next, starting from point U_m^0, "we walk" in the direction of j and then in the direction of i (lexicographic order) and calculate each U_i^j by using (A.19). At each step, conditions (A.27) – (A.29) are tested to check if the free border has been reached. The procedure is repeated until the values of U_i^j converge to the solution.

The figure below presents a pseudo-code of the routine implemented in Matlab.

U´ indicates “new” value (calculated in the current interaction)
U indicates “old” value (calculated in the previous interaction)

Initializes variables and defines constants

While error >epsilon

For i= 1 to My-1
For j=1 to Mx-1

$$\hat{U}_i^j = \frac{1}{\beta_{i\,j}^0}\left(\Delta y^2\, I - \beta_{i\,j}^1 U_i^{j+1} - \beta_{i\,j}^2 U'^{\,j-1}_i - \beta_{i\,j}^3 U_{i+1}^j - \beta_{i\,j}^4 U'^{\,j}_{i-1}\right)$$

$$U'^{\,j}_i = \mathrm{Max}\left(U_i^j + w(\hat{U}_i^j - U_i^j), EI\right)$$

end
end

error = ||**U´- U**||
end

Figure A.3
Pseudo-code for the solution of the (A1) – (A6) problem U' indicates “new” value (calculated in the current interaction) U indicates “old” value (calculated in the previous interaction)

Appendix 2

This appendix presents the solution for equation (18). The method involves a discrete approximation of the *value matching* and *smooth pasting* conditions (13) and (14) for determining A_1 and P_H. For a description of an application in a context similar to that of this article, see Dixit and Pindyck (1994).

First, we replace (18) in the both conditions and then divide (13) by (14). Then

$$V_1(P_H, K) = \frac{P_H}{\beta_1} \frac{\partial V_1(P_H, K)}{\partial P} \tag{A.30}$$

Then the transformation of variables $y \equiv \ln P$; $x \equiv 2\sqrt{K}$ and $F(y, x) \equiv V_1(P, K)$ is performed. Thus,

$$F(y^*, x) = \frac{1}{\beta_1} \frac{\partial F_1(y^*, x)}{\partial y} \tag{A.31}$$

Next, a discrete approximation to the derivative in the same grid used in Appendix 1 is defined,

$$\frac{\partial F(y^*, x)}{\partial y} = \frac{U_{i+1}^j - U_i^j}{\Delta y} \tag{A.32}$$

and (A.31) is determined using (A.32)

$$U_i^j = \frac{U_{i+1}^j}{1 + \beta_1 \Delta y} \tag{A.33}$$

With the $V_1(P, K)$ values obtained from the solution of equation (22), we start from point U_m^0 and "go downwards" in the grid, testing at each point the validity of (A.33). In a discrete environment, it is accepted that (A.33) has value within an error margin, that is,

$$U_i^j - \frac{U_{i+1}^j}{1 + \beta_1 \Delta y} < \text{eps} \tag{A.34}$$

Starting from the point where (A.34) occurs, (13) is used to determine A_1 and (18) to calculate the value of $V_0(P, K)$ for the interval $P < P_H$.

Appendix 3

This appendix presents, based on Dixit and Pindyck (1994), the economic intuition of *value matching* and *smooth pasting* conditions. A demonstration is made for the equation (11) but it may be extended to the other problems of the article. The following demonstrations are not strict mathematical proofs, providing only the economic intuition of the conditions.

Value Matching

The demonstration here is made by contradiction. In the problem of the company while inactive provided by the Bellman equation (8), the company maximizes its value and chooses whether to stay inactive or begin the construction of the asset. There is, in the solution of this problem, a price P_H that establishes the price above which the value of the construction-phase company is higher than that of the inactive company, and below which the contrary occurs. Consequently, to maximize its value, the company needs to establish this P_H. To make it more clear, the P_H must be such that, for $P < PH$, $V_0(P,K) > V_1(P,K)$ and for $P > P_H$, $V_0(P,K) < V_1(P,K)$. The *value matching* condition affirms that, in P_H, $V_0(P_H,K) = V_1(P_H,K)$. First, assume that this does not occur and $V_0(P_H,K) < V_1(P_H,K)$. Through the continuity[4] of the functions V_0 and V_1, for P slightly lower than P_H and with a sufficiently small dt, $V_0(P,K) \leq V_1(P,K)$ is valid, which contradicts the fact that $V_0(P,K) > V_1(P,K)$ occurs when $P < P_H$. On the other hand, if $V_0(P_H,K) > V_1(P_H,K)$, for P slightly higher than P_H, $V_0(P_H,K) \geq V_1(P_H,K)$ is valid, which contradicts the fact that $V_0(P,K) < V_1(P,K)$ when $P > P_H$. Consequently, if P_H is the boundary price, then $V_0(P,K) = V_1(P,K)$.

Smooth Pasting

Again, the proof is given by contradiction. Figure A.4 below helps in the demonstration.

First, consider that $\partial V_0/\partial P > \partial V_1/\partial P$, as in part a) of Figure A.4. Should this occur, it would be better for the company to continue to be inactive in P_H, because if $P > P_H$, it will occur that $V_0(P,K) > V_1(P,K)$ (the dotted line versus the full line), which contradicts the fact that P_H is the boundary price.

If $\partial V_0/\partial P < \partial V_1/\partial P$, as in part (b) of Figure A.4, it will be necessary to resort to the properties of the Wiener movement to show the contradiction. First, consider that, instead of exercising the option to invest, as should occur in P_H, the company adopts the following strategy: it does not exercise the option now and if, in the immediately following moment dt, $dP > 0$ should occur, it exercises the option and becomes a construction-phase company; and if $dP < 0$ occurs, it continues inactive. The expected present value of this strategy is

[4] Although a continuity hypothesis is not explicitly mentioned in the dissertation, it is assumed throughout the paper.

$$(1-rdt)\left[\varphi(dP<0)V_0(P+dP,K+dK)+\varphi(dP>0)V_1(P+dP,K+dK)\right] \tag{A.35}$$

where $\varphi(dP<0)$ is the probability of dP being lower than zero and $\varphi(dP>0)$ of it being higher.

Note that, if it is not considered that the strategy's future value should be discounted to present (given by factor 1-rdt in equation A.35), the alternative strategy is higher because by assumption $\partial V_0/\partial P < \partial V_1/\partial$P (see part (b) of the figure and compare the dotted line with the full line). What guarantees that the alternative strategy is higher, even discounted to present, is the fact that for sufficiently small dt, dP is of the order $\sqrt{dt}$, since the Wiener movement is of the order $\sqrt{dt}$, whereas the discount factor $(1-rdt)$ is of the order $dt < \sqrt{dt}$. Consequently, the strategy of not exercising immediately the investment option is better than the strategy of exercising it, which contradicts the fact of P_H is the boundary price.

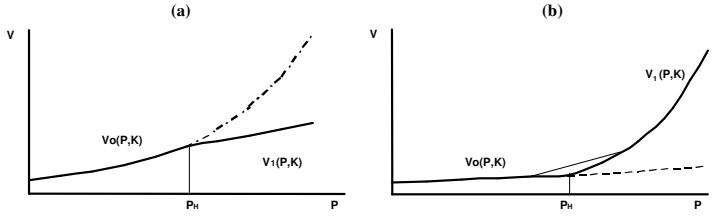

Figure A.4
Note that the we assume that the value matching condition is valid