

Computing Conditional VaR Using Time-varying Copulas

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Abstract

It is now widespread the use of the Value-at-Risk (VaR) as a canonical measure of risk. Most accurate VaR measures make use of some volatility model such as GARCH-type models. However, the pattern of the volatility dynamic of a portfolio follows from the (univariate) behavior of the risky assets, as well as from the type and strength of the associations among them. Moreover, the dependence structure among the components may change conditionally to past observations. Some papers have attempted to model this characteristic by assuming a multivariate GARCH model, or by considering the conditional correlation coefficient, or by incorporating some possibility for switches in regimes. In this paper we address this problem using time-varying copulas. Our modeling strategy allows for the margins to follow some FIGARCH type model while the copula dependence structure changes over time.

Resumo

O Valor-em-Risco (VaR) é hoje certamente a medida mais utilizada na mensuração do risco. As estimativas mais acuradas do VaR são aquelas baseadas em modelos de volatilidade tais como os modelos da família GARCH. Contudo, o padrão da dinâmica da volatilidade de uma carteira de investimentos depende não só do comportamento marginal dos ativos componentes, mas também do tipo e grau da associação entre os mesmos. Mais ainda, a estrutura de dependência entre esses componentes pode mudar com o tempo, condicionalmente às observações conjuntas passadas. Alguns artigos já consideraram este tópico tratando-o através de uma modelagem GARCH multivariada, ou considerando o coeficiente de correlação condicional, ou incorporando a possibilidade de mudanças de regime. Neste artigo tratamos este problema usando cópulas com parâmetros variando no tempo. Nossa estratégia de modelagem inclui a modelagem univariada através dos modelos FIGARCH, enquanto que a estrutura de dependência é modelada por uma cópula condicional aos valores passados.

Keywords: conditional copulas; FIGARCH models; conditional value-at-risk.

JEL Codes: C32; C50.

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1. Introduction

According to the Capital Adequacy Directive of the Basel Committee, the Value-at-Risk of a portfolio is a value large enough to cover its losses over a N -day holding period with a probability of $(1 - \alpha)$ (denoted by $\text{VaR}(\alpha; N)$, usually $\alpha = 0.01$ and $N = 10$ days). Despite this very simple definition, its accurate estimation may be not so simple, since it is highly dependent on the correct specification of the multivariate probability distribution of the variables composing the portfolio. We note that while the $\text{VaR}(\alpha; N)$ estimate is held fix during the period of 10 days, each component and/or the dependence structure connecting them, keep varying with time. This concern is the subject of this paper.

Series of financial log-returns present time-varying moments which may be modeled by some combination of ARFIMA and FIGARCH type models. As these conditional models provide better forecasts, conditional risk measures were introduced and are now standard tools in finance, for example, the conditional VaR.

However, the pattern of the volatility dynamic of a portfolio follows from the (univariate) behavior of the component asset returns, as well as from the type and strength of the associations among them. Moreover, the dependence structure among the components may change over time, conditionally to past observations. Some papers have attempted to model this characteristic by assuming a multivariate GARCH model; or by considering the conditional correlation coefficient, that is, the correlation coefficient based on just the very large (or very small) observations; or by assuming regime switches which would incorporate correlation breakdowns associated with economic downturns. Besides the VaR estimation, the specification of the multivariate conditional distribution is the basis for many important financial applications, for example, portfolio selection, option pricing, asset pricing models. In this paper we specify the multivariate conditional distribution of asset returns using copulas.

A d -dimensional copula is a cumulative distribution function (cdf) in $[0, 1]^d$ with uniform $(0, 1)$ margins. It summarizes the dependence structure independently of the specification of the marginal distribution. We shall see later on in this paper the advantages of this definition, which allows us to properly model the each margin.

Copulas have become standard tools in fields of finance and insurance (see Georges et al. (2001), Embrechts et al. (2003), Cherubini et al. (2004), Fermanian and Scaillet (2004), among others). Applications using dynamic copulas were proposed more recently. Patton (2001) introduced the conditional copula in the bivariate case. Modeling exchange rates, he assumed a bivariate Gaussian conditional copula with the correlation coefficient following a GARCH-type model. He considered also structural breaks and asymmetric copulas. Similarly, Genest et al. (2003) allowed the Kendall's correlation coefficient to evolve through time according current values of the conditional marginal variances.

Fermanian and Scaillet (2004) introduced the concept of *pseudo-copulas*. They showed that the copula models defined in Patton (2001), Rockinger and Jondeau

(2001), and also in Genest et al. (2003) are all pseudo-copulas. They proposed a nonparametric estimator of the conditional pseudo-copulas, derived its normal asymptotic distribution, and built up a goodness of fit test statistics.

Time varying dependence structure was also considered by Van Den Goorbergh et al. (2005) for modeling the relation between bivariate option prices and the dependence structure of the underlying financial assets. Patton (2003) found time variation to be significant in a copula model for asymmetric dependence between two exchange rates where the dependence parameter followed an ARMA-type process.

In this paper we address the problem of modeling the evolution through time of the bivariate distribution of financial log-returns using conditional pseudo-copulas. Considering the t -student copula, we achieve great flexibility by allowing the correlation coefficient and the number of degrees of freedom to vary over time according to the previous bivariate realizations. The model has the potential of providing more accurate estimation and forecasting of the joint behavior of risky assets, since it discriminates between stressful times and usual times, as well as joint positive and joint negative returns.

We define conditioning subsets of the $[0, 1]^2$, which may be related to several lagged values. Theoretical aspects are in Doukhan (2004). Testing if the dependence structure really depends on past values is a very important issue. We test the constancy of copula parameters using tests developed in Fermanian and Wegkamp (2004).

In Section 2 we provide a brief review of copula definitions, introduce our models, and give some theoretical support for them. In Section 3 we report results from a small simulation experiment to assess models suitability for capturing the evolution of copula parameters. In Section 4 we present an application where we compute the conditional VaR. In Section 5 we conclude and discuss some ideas for further research.

To simplify the notation, in what follows we give results and models in the bivariate case.

2. Conditional Copulas

Consider a stationary process $(X_{1,t}, X_{2,t})_{t \in \mathcal{Z}}$. In the case the joint law of $(X_{1,t}, X_{2,t})$ is independent of t , the dependence structure is of (X_1, X_2) given by its (constant) copula C . If (X_1, X_2) is a continuous random vector with joint cdf F and marginals F_1, F_2 , then, there is a unique copula C pertaining to F , defined on $[0, 1]^2$ such that

$$C(F_1(x_1), F_2(x_2)) = F(x_1, x_2)$$

holds for any $(x_1, x_2) \in \mathbb{R}^2$.

However, in many situations, there exists some time dependent structure, which may be captured by conditional distributions with respect to past observations. In the multivariate setting we could consider conditional copulas. A time dependent copula may not satisfy all properties of a (true) copula (see Nelsen

(1999)). The concept of *pseudo-copulas*, introduced by Fermanian and Scaillet (2004), formalize and unify some previous attempts in the direction of modeling time varying dependence structures using copulas.

A *2-dimensional pseudo-copula* is a function $C : [0, 1]^2 \mapsto [0, 1]$ satisfying all copula properties, except that $C(u, 1)$ is not necessarily equal to u (or $C(1, v)$ is not necessarily equal to v). Fermanian and Scaillet (2004) proved the equivalent of the Sklar's theorem for a pseudo-copula.

Now, let (X_1, X_2) be a continuous random vector from $(\Omega, \mathcal{A}_0, P)$ to \mathbb{R}^2 , and let $\mathcal{A}_1, \mathcal{A}_2$ and \mathcal{B} be some arbitrary sub- σ -algebras.¹ Theorem 3 in Fermanian and Scaillet (2004) states that there exists a random function $C : [0, 1]^2 \times \Omega \rightarrow [0, 1]$ such that

$$P\{(X_1, X_2) \leq (x_1, x_2) | \mathcal{B}\}(w) = C(P\{X_1 \leq x_1 | \mathcal{A}_1\}(w), \\ P\{X_2 \leq x_2 | \mathcal{A}_2\}(w), w) \equiv C(P\{X_1 \leq x_1 | \mathcal{A}_1\}, P\{X_2 \leq x_2 | \mathcal{A}_2\})(w),$$

for every $(x_1, x_2) \in \mathbb{R}^2$ and almost every $w \in \Omega$. This function C is $\mathcal{B}([0, 1]^2) \otimes \sigma(\mathcal{A}_1, \mathcal{A}_2, \mathcal{B})$ measurable. For almost every $w \in \Omega$, $C(\cdot, w)$ is a pseudo-copula and is uniquely defined on the product of the values taken by $x_j \mapsto P\{X_j \leq x_j | \mathcal{A}_j\}(w)$, $j = 1, 2$. We use the notation $C(\cdot | \mathcal{A}_1, \mathcal{A}_2, \mathcal{B})$ in the case C is the unique $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{B})$ -pseudo copula associated with (X_1, X_2) .

Note that $C(\cdot | \mathcal{A}_1, \mathcal{A}_2, \mathcal{B})(w)$ may not be a copula, because in general, the information provided by \mathcal{B} and \mathcal{A}_1 , or by \mathcal{B} and \mathcal{A}_2 is not the same. In general, the law of (X_1, X_2) conditional on \mathcal{B} does not provide information on the conditional marginal laws $F_j(\cdot | \mathcal{A}_j)$, $j = 1, 2$. $C(\cdot | \mathcal{A}_1, \mathcal{A}_2, \mathcal{B})(w)$ is a true copula if and only if

$$P\{X_j \leq x_j | \mathcal{B}\} = P\{X_j \leq x_j | \mathcal{A}_j\}, \text{ almost everywhere}$$

for $j = 1, 2$ and all $(x_1, x_2) \in \mathbb{R}^2$. This means that \mathcal{B} cannot provide more information about X_j than \mathcal{A}_j , for every $j = 1, 2$. For example, when $\mathcal{B} = \mathcal{A}_1 = \mathcal{A}_2$, such as in Patton (2001), we have a true conditional copula.

When modeling time dependent data, the σ -algebras $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{B})$ are usually indexed by t : $\mathcal{A}_{j,t} = \sigma(X_{j,t-1}, X_{j,t-2}, \dots)$, $j = 1, 2$, and $\mathcal{B}_t = \sigma((X_{1,t-1}, X_{2,t-1}), (X_{1,t-2}, X_{2,t-2}), \dots)$. The conditional and pseudo copulas depend thus on the index t and on the past values $(X_{1,t-1}, X_{2,t-1})$, $(X_{1,t-2}, X_{2,t-2}), \dots$ of (X_1, X_2) , being a sequence of copulas. In this paper we study two cases:

1. $\mathcal{A}_{j,t} = (X_{j,t-1} \in [a_j, b_j])$ for some $a_j, b_j \in \mathbb{R}$, $a_j < b_j$, $j = 1, 2$, and $\mathcal{B}_t = ((X_{1,t-1}, X_{2,t-1}) \in [a_1, b_1] \times [a_2, b_2])$.

¹In fact there is one restriction on these sub- σ -algebras, which are satisfied if the marginal conditional cdf's, $F_j(x_j | \mathcal{A}_j) = P(X_j \leq x_j | \mathcal{A}_j)$, $j = 1, 2$, are strictly increasing, as we assume in this paper.

2. $\mathcal{A}_{j,t} = ((X_{j,t-1} \in [a_j, b_j]), (X_{j,t-2} \in [c_j, d_j]))$ for some $a_j, b_j, c_j, d_j \in \mathbb{R}$, $a_j < b_j, c_j < d_j, j = 1, 2$, and $\mathcal{B}_t = ((X_{1,t-1}, X_{2,t-1}) \in [a_1, b_1] \times [a_2, b_2]), (X_{1,t-2}, X_{2,t-2}) \in [c_1, d_1] \times [c_2, d_2])$.

Our application is in the field of finance. Even though conditional univariate GARCH-type models, widely used in practice, usually adjust properly to log-returns data, there are many situations where a (conditional) dependence structure must be specified. For example, in portfolio optimization. This motivated Rockinger and Jondeau (2001) to develop a methodology for measuring conditional dependence by assuming time-varying copulas and GARCH-type models with time-varying skewness and kurtosis in the marginal distributions. Using the Hansen's (1994) generalized t-student as the error distribution for the GARCH models and the Plackett's copula, they provided empirical evidence that the dependency between financial returns may evolve through time.

Likewise Rockinger and Jondeau (2001), here we assume a parametric pseudo-copula C_θ conditional to the position of past joint observations in the unit square. In the first model, corresponding to the sub- σ -algebras (1), we decompose the unit square into 16 squares, denoted by $\mathcal{S}_j, j = 1, \dots, 16$. That is, $\mathcal{S}_1 = [0, 1/4] \times [0, 1/4]$; $\mathcal{S}_2 = (1/4, 2/4] \times [0, 1/4]$; ...; and $\mathcal{S}_{16} = (3/4, 1] \times (3/4, 1]$. We define 16 possibilities for the parameter θ , denoted by $\theta_{\mathcal{S}_1}, \theta_{\mathcal{S}_2}, \dots, \theta_{\mathcal{S}_{16}}$, according to: θ_t is $\theta_{\mathcal{S}_1}$ whenever $(u_{t-1}, v_{t-1}) \in \mathcal{S}_1$; ...; and θ_t is $\theta_{\mathcal{S}_{16}}$ whenever $(u_{t-1}, v_{t-1}) \in \mathcal{S}_{16}$. In this paper we use the Gaussian($\theta = \rho$) and the t -student($\theta = (\rho, \nu)$) parametric families of copulas (see Demarta and McNeill (2004) for the definition and expressions related to the t -copula). In the case of the t -copula, the dynamic behavior of the copula parameters obeys

$$\theta_t = (\rho_t, \nu_t) = \sum_{j=1}^{16} (\rho_{\mathcal{S}_j}, \nu_{\mathcal{S}_j}) \mathbf{I}_{[(u_{t-1}, v_{t-1}) \in \mathcal{S}_j]} \quad (1)$$

where $\mathbf{I}_{[E]}$, is the indicator function of event E . This model is asymmetric (for example, $\theta_{\mathcal{S}_3}$ is not necessarily equal to $\theta_{\mathcal{S}_9}$, or $\theta_{\mathcal{S}_1}$ is not necessarily equal to $\theta_{\mathcal{S}_{16}}$) and allows for testing various interesting hypothesis on the $\theta_{\mathcal{S}_j}$ values, including equality among some of them, zero correlation for some j , constant correlation, and the effect of returns' sign and magnitude on the subsequent strength of dependence (effect of *joint bad or joint good news*).

We extend this model and let the copula parameters on time t to depend on the past values $((X_{1,t-1}, X_{2,t-1}), (X_{1,t-2}, X_{2,t-2}))$, according to σ -algebra (2). Now there are 256 possibilities for θ according to the location of $((u_{t-1}, v_{t-1}), (u_{t-2}, v_{t-2}))$ in the 16×16 squares, denoted by $\mathcal{S}_{i,j}, i, j = 1, \dots, 16$. For the t -copula we have:

$$\theta_t = (\rho_t, \nu_t) = \sum_{i=1}^{16} \sum_{j=1}^{16} (\rho_{\mathcal{S}_{i,j}}, \nu_{\mathcal{S}_{i,j}}) \mathbf{I}_{[(u_{t-1}, v_{t-1}) \in \mathcal{S}_i]} \mathbf{I}_{[(u_{t-2}, v_{t-2}) \in \mathcal{S}_j]}. \quad (2)$$

Besides the characteristics already mentioned for model (1), model (2) allows for testing many combinations of events of interest, for example, significance and/or equality of the parameters $\theta_{S_{1,1}}$ and $\theta_{S_{16,16}}$. A drawback is that the number of observations per area now will be much smaller.

To estimate, we apply the maximum likelihood method in two steps (see Genest and Rivest (1993), Shi and Louis (1995), and also Chebrian et al. (2002)). We first fit the FIGARCH model to the log-returns, obtain the residuals, and then we apply the probability integral transformation using the estimated conditional distribution to obtain the *uniform*(0, 1) data. That is, we obtain

$$u_t = F_1(x_{1,t}, \sigma_{1,t}^2 | \mathcal{A}_{1,t}) \quad (3)$$

$$v_t = F_2(x_{2,t}, \sigma_{2,t}^2 | \mathcal{A}_{1,t}), \quad (4)$$

which are used to estimate the copulas.

For example, for model (1), we maximize with respect to $\theta_{S_1}, \theta_{S_2}, \dots, \theta_{S_{16}}$ the log-likelihood (\mathcal{LL})

$$\mathcal{LL} = \sum_{j=1}^{16} \sum_{t=2}^T \log c(u_t, v_t, \rho_{S_j}, \nu_{S_j}) \mathbf{I}_{[(u_{t-1}, v_{t-1}) \in \mathcal{S}_j]} \quad (5)$$

where T is the sample size. Note \mathcal{LL} equals the sum $\sum_{j=1}^{16} \mathcal{LL}_j$ where \mathcal{LL}_j is the log-likelihood computed using data following the event $[(u_{t-1}, v_{t-1}) \in \mathcal{S}_j]$.²

The time path for the upper (lower) tail dependence coefficient³ is obtained by plugging the paths obtained for ρ_j and ν_j in the formula of λ_U (λ_L).

The models allow for testing some nested statistical hypothesis using the log-likelihood ratio chi-square statistic. We test the constancy of the parameters of the conditional pseudo-copulas with respect to t , that is, the null hypothesis \mathcal{H}_0 : $\rho_{S_1} = \rho_{S_2} = \dots = \rho_{S_{16}}$ against model (1), using the chi-square test statistic (15 degrees of freedom for fixed ν). We also test if the model is asymmetric, that is, \mathcal{H}_0 : $\rho_{S_1} = \rho_{S_{16}}$ versus the alternative $\rho_{S_1} > \rho_{S_{16}}$, we compare \mathcal{LL}_1 and \mathcal{LL}_{16} . We test if returns react differently to joint large (positive or negative) or joint small (positive and negative) realizations, by testing \mathcal{H}_0 : $\rho_{S_1} = \rho_{S_6}$ versus the alternative $\rho_{S_1} > \rho_{S_6}$, and \mathcal{H}_0 : $\rho_{S_{11}} = \rho_{S_{16}}$ versus the alternative $\rho_{S_{16}} > \rho_{S_{11}}$.

²It would be interesting to also experiment the robust estimation procedures proposed by Mendes et al. (2005), since they emphasize data in specific regions of $[0, 1]^2$. Since we are using here elliptical pseudo-copulas, we expect good results using the weighted maximum likelihood estimates. We leave this for future work.

³The coefficients of upper and lower tail dependence are given by

$$\lambda_U = \lim_{u \uparrow 1} \frac{\overline{C}(u, u)}{1 - u}, \text{ where } \overline{C}(u, v) = \Pr\{U > u, V > v\} \text{ and } \lambda_L = \lim_{u \downarrow 0} \frac{C(u, u)}{u}.$$

provided a limit $\lambda_U \in [0, 1]$ exists.

3. Simulations

We now report the results from two simulation experiments, designed to assess the performance of the proposed models. In our experiments we do not address the problem of misspecification of the copula family.

Experiment 1: *True model is Gaussian constant on time (ρ fixed) but the time varying model (1) is assumed and estimated.*

We generate bivariate data (sample size 1500) based on a Gaussian copula with ρ fixed and equal to 0.70, and GARCH(1,1) margins. Using the *IFM* method we estimate the dynamic model (1) based on a Gaussian copula. For each of the 200 repetitions we tested and did not reject the equality of all 16 correlation coefficients (with very few exceptions). The overall mean and 95% confidence interval of the ρ_{S_j} values, $j = 1, \dots, 16$, are, respectively, 0.7014, [0.6109, 0.7715]. Using the constant copula we obtained the mean 0.6996, and the 95% confidence interval of [0.6849, 0.7143]. The conclusion is that a misspecified time varying model may indicate the data follow a constant in time model.

Experiment 2: *True model is some time-varying Gaussian, and model (1) and a constant copula are estimated.*

In this experiment we mimic a situation where the strength of dependence depend upon volatility (or magnitude of absolute value data). The true data generating process is Gaussian such that, on time t , conditionally on the previous observations (u_{t-1}, v_{t-1}) fall in squares S_1 or S_{16} , $\rho = 0.9$; in squares S_6, S_7, S_{10}, S_{11} , $\rho = 0.7$; in squares S_4, S_{13} , $\rho = 0.5$; (4) squares $S_2, S_3, S_5, S_8, S_9, S_{12}, S_{14}, S_{15}$, $\rho = 0.3$. The number of simulations is 200, and the constant copula mean estimate is 0.602, with standard deviation of 0.014. Model (1) performed very well and yielded, for each square, correlation coefficients means and standard deviations given in Table 1.

Table 1
Correlation coefficients means and standard errors from Experiment 2.

Square	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
True ρ	0.9	0.3	0.3	0.5	0.3	0.7	0.7	0.3
$\hat{\rho}$	0.900	0.309	0.274	0.488	0.299	0.704	0.701	0.298
$s.e.(\hat{\rho})$	0.007	0.061	0.122	0.108	0.079	0.024	0.031	0.117
Square	S_9	S_{10}	S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	S_{16}
True ρ	0.3	0.7	0.7	0.3	0.3	0.9	0.3	0.9
$\hat{\rho}$	0.288	0.698	0.702	0.293	0.517	0.276	0.303	0.900
$s.e.(\hat{\rho})$	0.108	0.028	0.025	0.079	0.091	0.124	0.083	0.007

4. Computing the Conditional VaR

To illustrate the usefulness of the time-varying copula model we use daily log-returns from the main indexes of two Latin America stock markets, the Argentinian (X_1) and the Brazilian (X_2) markets. The period covered is January 1st, 1994 to January 31st, 2005.

We assume $\{(X_{1,t}, X_{2,t})\}_{t=1}^T$ is a stationary process. The general model may be written as:

$$(X_{1,t}, X_{2,t}) = (\mu_{1,t}, \mu_{2,t}) + \sqrt{\Sigma_t(\epsilon_{1,t}, \epsilon_{2,t})'} \quad (6)$$

where, conditional to \mathcal{I}_t , where \mathcal{I}_t denote the information set at time t , $(\mu_{1,t}, \mu_{2,t})$ is the true expectation of $(X_{1,t}, X_{2,t})$ and Σ_t is a diagonal matrix with elements $(\sigma_{1,t}, \sigma_{2,t})$. The sequence of standardized bivariate innovations $\{(\epsilon_{1,t}, \epsilon_{2,t})\}_{t=1}^T$ are independent of \mathcal{I}_t , *i.i.d.* with zero means and unit variances. The innovations $(\epsilon_{1,t}, \epsilon_{2,t})$ possess copula C and univariate cdfs F_1 and F_2 .

The conditional means $(\mu_{1,t}, \mu_{2,t})$ and conditional variances $(\sigma_{1,t}^2, \sigma_{2,t}^2)$ specifications will be drawn from the ARMA and FIGARCH⁴ families. To each margin we fit by maximum likelihood a wide selection of combination of these models, and choose the best one according to the AIC criterion followed by all necessary statistical tests for verification of parameters estimates significance and check of models assumptions. The residuals are used to obtain the (u_t, v_t) data, which were tested with respect to independence and the *uniform*(0, 1) assumption (Shapiro test for normality). Figure 1 shows the bivariate data of log-returns at left, and the corresponding standardized uniform(0, 1) data at the right hand side.

The parametric copula specifications are those given in Section 2, namely the Normal (or Gaussian) copula and the t-copula. We recall that in the case of constant ρ , the t-copula will have symmetric tail dependence. The smaller the number of degrees of freedom, the greater the tail dependence, and the higher the probability of joint extreme events.

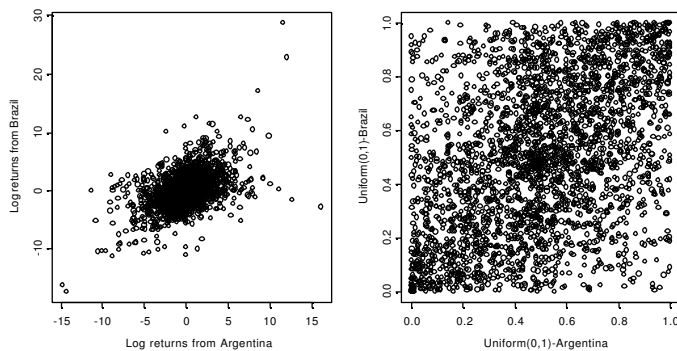


Figure 1

The bivariate data of log-returns at left, and the corresponding standardized uniform(0, 1) data at the right hand side

⁴We provide in the Appendix a review on FIGARCH models.

0.29 (13)	0.34 (13)	0.45 (9)	0.43 (6)
S13	S14	S15	S16
0.5 (13)	0.46 (9)	0.48 (10)	0.35 (7)
S9	S10	S11	S12
0.5 (13)	0.37 (10)	0.5 (8)	0.33 (13)
S5	S6	S7	S8
0.56 (12)	0.51 (9)	0.42 (13)	0.27 (3)
S1	S2	S3	S4

Figure 2
Correlation coefficient (and degrees of freedom) estimates at each S_j from model (1)

We start by fitting model (1) to the transformed data. Figure 2 shows the correlation coefficient (and degrees of freedom) estimates at each S_j . We observe that dependence is higher when at least one index is extreme and negative in the previous day (S_1, S_2, S_5). The test of the constancy of the parameters over time, $\mathcal{H}_0 : \rho_{S_1} = \rho_{S_2} = \dots = \rho_{S_{16}}$, rejected the null with a p-value of 0.006 (the constant t -copula fit resulted in $\rho = 0.45$, $\nu = 10$, and $\lambda_L = \lambda_U = 0.0683$). The null $\mathcal{H}_0 : \rho_{S_1} = \rho_{S_{16}}$ was also strongly rejected against the alternative $\rho_{S_1} > \rho_{S_{16}}$. This means that the markets react differently to joint extreme negative or positive previous observations, an well known type of asymmetry (information asymmetry or effect of bad news). The Ljung-Box test applied to the estimated path of ρ_t rejected the null hypothesis of zero autocorrelation.

The fit of model (2) resulted in correlation coefficient values ranging from -0.87 ($S_{9,12}$) to $+0.92$ ($S_{1,4}$). However, the log-likelihood ratio test did not reject the simpler time-varying model in favor of the complex time-varying model. Thus we try another model, model (3), which is a simplification of model (2).

Model (3) assumes that ν is fixed and equal to the value found for the constant copula ($\nu = 10$), and that the correlation coefficients on time t depend on data on times $t - 1$ as in models (2) and (3), and on times $t - 2$ according to which quadrant the joint observations (u_{t-2}, v_{t-2}) fall. These quadrants are S_1^* , representing the $(--)$ data, S_2^* representing the $(+-)$ observations, and S_3^* and S_4^* corresponding respectively to the $(-+)$ and the $(++)$ data. There are thus $16 \times 4 = 64$ possibilities for the correlation coefficient.

$$\theta_t = \rho_t = \sum_{j=1}^4 \sum_{i=1}^{16} \rho_{S_{ij}} \mathbf{I}_{[(u_{t-1}, v_{t-1}) \in S_i]} \mathbf{I}_{[(u_{t-2}, v_{t-2}) \in S_j^*]}. \quad (7)$$

The log-likelihood ratio test strongly rejected model (1) in favor of model (3). Several characteristics of the dependence structure may be inferred from the estimates. For example, the estimate of the correlation coefficient $\rho_{S_{1,1}}$ of data following extreme negative returns during two consecutive days is 0.57. Note that the value of the tail dependence coefficient under the constant copula model is 0.068, and thus the strenght of interdependence during stressful periods would be higher if estimated using dynamic models. When the the past joint observations are in $S_{16,4}$ (joint consecutive extreme positive) the estimate is smaller, 0.47. This reveals the asymmetry of the dependence structure.

Figure 3 shows the evolution through time of the correlation and tail dependence coefficients, during the most recent 100-days period. In the first row we show the $\hat{\rho}$ path, and in the second row the path of $\hat{\lambda}$. They should be compared to the constant copula results: $\hat{\rho} = 0.45$ and $\hat{\lambda} = 0.068$.

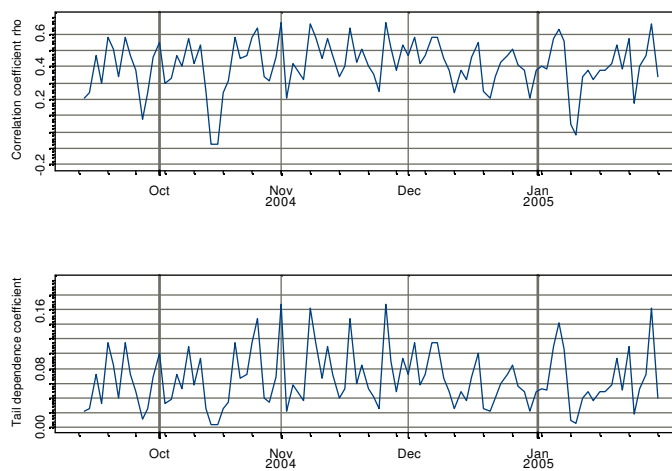


Figure 3
Evolution through time of the copula correlation and tail dependence coefficients, during the most recent 100-days period, and assuming copula model (3)

One application of the proposed models is the computation of the conditional Value-at-Risk. Suppose a portfolio is composed by d different instruments, with nominal amount w_i invested into asset i , $i = 1, \dots, d$. Assume that there is no temporal dependence in the portfolio components series, and that they follow a multivariate normal distribution (constant Gaussian copula). In this case all uni-

variate margins are normal with standard deviation σ_i and pairwise correlation coefficients given by ρ_{ij} , $i, j = 1, \dots, d$. The RiskMetrics formula for the $\text{VaR}(\alpha; N)$ is

$$\text{VaR}(\alpha; N) = z_\alpha \sigma \sqrt{N} \quad (8)$$

where z_α is the α -quantile of the standard normal distribution, and σ is the portfolio standard deviation, the square root of

$$\sigma^2 = \sum_{i=1}^d w_i^2 \sigma_i^2 + \sum_{i \neq j} w_i w_j \sigma_i \sigma_j \rho_{ij} \quad (9)$$

Now suppose $d = 2$ and $N = 1$. In this case, formula (8) becomes

$$\text{VaR}(\alpha) = \sqrt{(\text{VaR}_1)^2 + (\text{VaR}_2)^2 + 2\rho \text{VaR}_1 \text{VaR}_2} \quad (10)$$

where VaR_j represents the $\text{VaR}(\alpha)$ of asset $j = 1, 2$, that is, $\text{VaR}_j = z_\alpha w_j \sigma_j$, and ρ is the correlation coefficient. We compute (10) using the previous fits: fixed-time and varying-time (model 3) copula parameters. We do that for the 1000 days at the end of the series. We estimate the models and compute the VaR one-step ahead, roll the window over the data, and repeat the process.

To decide which procedure provided a more accurate VaR estimate at the 1% level, we observe the number of times that the series fell beyond the VaR. Model 3 performed better (11 times against 13 for fixed), the expected is 10. The expected shortfall is -1.5462 and -1.6296 , respectively for VaR computed using constant ρ and time-varying ρ , at the 1% risk. So, the expected loss given that the daily return is more extreme than the VaR value, is smaller under the fixed time model.

5. Conclusions and Discussions

In this paper we proposed a model for dynamically estimate the dependence structure of a set of financial returns. Our modeling strategy allows for the margins to follow some FIGARCH type model while the copula dependence structure also changes over time. An interesting feature of our model is that one can test the effect of some selected scenarios, for example, the effect of extreme joint (positive or negative) returns on the subsequent dependence among the returns. Our exposition was restricted to the bivariate case, but models can be easily implemented and run relatively fast in higher dimensions.

Research on time varying copulas is still in its infancy. There are many open questions, and there is room for both theoretical and computational developments. Many applications will naturally follow.

In this paper we provided empirical evidence that the dependence structure (given by its copula) among asset returns may be best represented by a time-varying copula. We computed the evolution in time of some linear and non linear measures of association. We showed that the Value-at-Risk estimation may

be improved by assuming a dynamic copula. In addition, dynamic estimation of the coefficient of tail dependence highlights the influence of previous joint observations, in particular joint negative returns, on the subsequent interdependence between the assets. These findings may be used by investors selecting portfolio components.

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Appendix

FIGARCH models

Among the so called *stylized facts* that characterize a return series, the behavior of the autocorrelation function (ACF) of the data and squared data deserves close attention. For the return series the sample ACF is typically negligible at almost all lags, except for the first and second ones (it decays exponentially). However, the sample ACF of the absolute values or their squares are all positive, usually decays slowly and tends to stabilize for large lags (hyperbolic decay rate). This empirical fact is usually interpreted as evidence of long memory in volatility.

The first long memory time series model proposed (for the mean) was the Fractionally Integrated ARMA model, the ARFIMA model, introduced by Granger and Joyeux (1980). An ARFIMA(p, d, q) process is a general class of processes for the mean which ranges from the unit root ARIMA($p, d = 1, q$) process, up to integrated processes of order 0. Perhaps the most theoretically discussed and empirically tested (Bollerslev and Mikkelsen (1999), Bollerslev and Wright (2000), Mikosch and Stărică (2003), among others) long range dependence class of volatility models consists of the Fractionally Integrated Generalized ARCH models, FIGARCH models, introduced by Baillie et al. (1996), and Bollerslev and Mikkelsen (1996). Other important alternative models are the Fractionally Integrated Stochastic Volatility models of Breidt et al. (1998), and the Two Component model of Ding and Granger (1996).

Let $\{r_t\}_{t=1}^T$ be a time series of asset returns. To capture the varying conditional variance of r_t it is assumed that

$$r_t = C + \varepsilon_t \quad (11)$$

where C is a constant and

$$\varepsilon_t | \mathcal{F}_{t-1} = \sigma_t z_t, \quad (12)$$

where z_t is an *i.i.d.* sequence of random variables with zero mean and unit variance, and \mathcal{F}_t represents the information set up to time t . According to Baillie et al. (1996) and Bollerslev and Mikkelsen (1996), a FIGARCH(r, d, s) model for the conditional variance σ_t^2 satisfies

$$\varepsilon_t^2 (1 - \phi(\mathcal{L})) (1 - \mathcal{L})^d = \omega + (1 - \beta(\mathcal{L})) (\varepsilon_t^2 - \sigma_t^2) \quad (13)$$

where $\omega > 0$ is a real constant, the *fractional integration parameter* $d \in [0, 1]$, \mathcal{L} is the lag operator, $\phi(\mathcal{L}) = \alpha(\mathcal{L}) + \beta(\mathcal{L})$, and $\beta(\mathcal{L}) = \sum_{j=1}^s \beta_j \mathcal{L}^j$. The *fractional difference operator* $(1 - \mathcal{L})^d$ can be expanded in a binomial series to produce an infinite polynomial in \mathcal{L} :

$$(1 - \mathcal{L})^d = 1 - \sum_{k=1}^{\infty} \delta_{d,k} \mathcal{L}^k = 1 - \delta_d(\mathcal{L}), \quad (14)$$

where the coefficients $\delta_{d,k} = d \frac{\Gamma(k-d)}{\Gamma(k+1)\Gamma(1-d)}$ in (14) are such that

$$\delta_{d,k} = \delta_{d,k-1} \left(\frac{k-1-d}{k} \right), \quad (15)$$

for all $k \geq 1$, where $\delta_{d,0} \equiv 1$.

The FIGARCH(r, d, s) process has the infinite ARCH representation:

$$\sigma_t^2 = \omega (1 - \beta(\mathcal{L}))^{-1} + \lambda(\mathcal{L}) \varepsilon_t^2, \quad (16)$$

where the polynomial $\lambda(\mathcal{L})$ is given by

$$\lambda(\mathcal{L}) = \sum_{k=0}^{\infty} \lambda_k \mathcal{L}^k = 1 - (1 - \beta(\mathcal{L}))^{-1} \phi(\mathcal{L}) (1 - \mathcal{L})^d. \quad (17)$$

A FIGARCH(r, d, s) processes must meet some parameters restrictions to ensure positivity of the conditional variance σ_t^2 .

Even though the series σ_t^2 is non-observable, its persistence properties are propagated to the observable series r_t^2 . Since the second moment of the unconditional distribution of r_t is infinite, the FIGARCH process is not weakly stationary. Discussions about stationarity property of FIGARCH processes may be found in Nelson (1988), Mikosch and Stăricu (2003), among others.

To assure the positiveness of the conditional variance, Bollerslev and Mikkelsen (1996) proposed the Fractionally Integrated Exponential GARCH (FIE-GARCH) model:

$$\phi(\mathcal{L})(1 - \mathcal{L})^d \ln \sigma_t^2 = w + \sum_{j=1}^r (\beta_j |\frac{\varepsilon_{t-j}}{\sigma_{t-j}}| + \gamma_j \frac{\varepsilon_{t-j}}{\sigma_{t-j}}), \quad (18)$$

where $\gamma_j \neq 0$ indicates the existence of leverage effects. By including the leverage term we allow the conditional variance to depend both on sign and magnitude of expected returns. This asymmetric model is an attempt to model another stylized fact about asset returns, the effect of *bad news*: risky stocks respond differently to positive high gains and low negative falls. The larger the leverage parameter value, the larger the risk.

In this paper we also considered the very interesting (FI)GARCH-in-mean model of Engle et al. (1987), which extends (11) to

$$r_t = C + \pi g(\sigma_t^2) + \varepsilon_t,$$

where $g(\cdot)$ can be an arbitrary function of the volatility, we use $g(\sigma_t^2) = \sigma_t^2$. This model captures the effect of volatility on expected returns. One of the rationales behind this model is the fact that a price fall reduces the value of an equity and then increases the debt-to-equity ratio, raising volatility.