Forecasting Exchange Rate Density Using Parametric Models: The Case of Brazil

Marcos M. Abe*
Eui J. Chang**
Benjamin M. Tabak***

Abstract
This paper employs a recently developed parametric technique to obtain density forecasts for the Brazilian exchange rate, using the exchange rate options market. Empirical results suggest that the option market contains useful information about future exchange rate density. These results suggests that density forecasts using options markets may add value for portfolio and risk management, and may be useful for financial regulators to assess financial stability.

***Keywords*:** Density forecasting; emerging market; exchange rate; options market.

JEL codes: G10; G15; F31.

Resumo
Este artigo emprega uma técnica paramétrica desenvolvida recentemente para extrair previsões de densidade para a taxa de câmbio doméstica, usando o mercado de opções cambiais. Os resultados empíricos sugerem que o mercado de opções contém informação útil sobre a densidade futura da taxa de câmbio. Estes resultados sugerem que previsões de densidade usando o mercado de opções podem adicionar valor à gestão de carteiras e de risco, e podem ser úteis para reguladores financeiros avaliarem estabilidade financeira.

***Palavras-chave*:** Previsão de densidade; mercados emergentes; taxa de câmbio; mercado de opções.

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*Banco Central do Brasil, Research Department (DEPEP). SBS Quadra 3, Bloco B, Ed. Sede, 9º andar, 70074-900, Brasilia, DF, Brazil. E-mail: Marcos.massaki@bcb.gov.br

**Banco Central do Brasil, Research Department (DEPEP). SBS Quadra 3, Bloco B, Ed. Sede, 9º andar, 70074-900, Brasilia, DF, Brazil. E-mail: eui.jung@bcb.gov.br

***Corresponding author. Tel. + 55 61 3414 3421. E-mail: Benjamin.Tabak@bcb.gov.br

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1. Introduction

Recent research in the financial literature has investigated whether option-implied distributions are useful in providing information regarding the future distribution of underlying asset prices (see Clews (2000) and Melick and Thomas (1997)). Many methods have been proposed in the literature to recover risk neutral density of financial assets (see Savickas (2002, 2004)), Rebonato (1999), Corrado (2001), Markose and Alentorn (2005), Dutta and Babbel (2002b,a), Gemmil and Saflekos (2000) and Melick and Thomas (1997)).[1]

The evaluation of density forecasts is also a topic of great importance for both portfolio and risk managers, financial regulators and in the insurance market. Recent literature has used several methods to evaluate density forecasts including Diebold et al. (1998), Clements and Smith (2000) and Elerian et al. (2001).

One of the most used distributions to extract density forecasts is the mixture of lognormals (see Ritchey (1990) and Melick and Thomas (1997)). These authors argue that the risk-neutral density of the asset price when options expire can be defined as a mixture of lognormal densities. The problem with these densities is that the number of parameters is large and overfitting problems may arise.

De Jong and Huisman (2000) study skewed student-t and compare their performance with non parametric methods, presenting evidence supporting parametric methods for extracting densities. Liu et al. (2003) study the FTSE-100 index and argue that parametric densities provide the most accurate predictive densities for real-world observed index levels. The authors compara GB2 densities with spline densities and find that GB2 densities have more explanatory power than historical densities. Dutta and Babbel (2002b) compare the performance of the g-and-h distribution with the GB2 for options on interest rates (LIBOR) and provide evidence in favor of the g-and-h distribution.[2]

Tunaru and Albota (2005) compare the performance of risk-neutral densities assuming a variety of methods: Weibull distribution, Generalized Gamma, GB2, Burr-3 and g-and h distributions. The authors focus on interest rates and find that GB2 perform quite well if compared to other distributions.

It is important to note that preliminary research suggests a variety of methods to extract risk-neutral densities. Some methods are more cumbersome than others and there is a trade-off between parsimony and accuracy. In general, models with more parameters yield more accurate estimates. However, these methods are in general more cumbersome and in some cases it is hard to calibrate the models. Therefore, models with a few parameters should be preferred whenever possible.

Many studies have been performed to assess the quality of density estimation for equity and exchange rate markets. However, the main focus of these studies has been on developed countries and very little research has studied emerging markets. The limited availability of data for emerging markets combined with

[1] See also Campos (2005).

[2] See also Aparicio and Hodges (1998), Bliss and Panigirtzoglou (2004), Panigirtzoglou and Skiadopoulos (2004), Shimko (1993), West and Cho (1995) and Taylor (2005).

underdeveloped derivatives markets is one of the main impediments for the development of research on these markets. This tries to reduce this gap by studying an emerging market, namely country-regionplaceBrazil, which has a liquid and a well-developed derivatives market for the domestic exchange rate.

Using data that covers the period from 2000 to 2005, the results of the study suggest that a parametric method, using the generalized beta density of second kind, is useful for density forecasting.

This paper argues for the use of the generalized beta density of second kind (GB2) for exchange rate returns in call option pricing models for the following reasons:

1. we have to estimate a small number of parameters, avoiding problems such as overfitting the data;
2. the parameters of the GB2 permit general combinations of the mean, variance, skewness and kurtosis, enabling the shape of the density to be flexible;
3. the real-world density has a closed form when one assumes the GB2 density, and 4;
4. recent literature suggests that the GB2 density forecasting accuracy performs quite well (see Tunaru and Albota (2005)).[3]

The remainder of the paper is organized as follows. Section 2 briefly presents the methodology. Section 3 describes the data and show empirical results. Section 4 concludes the paper.

2. Methodology

2.1 Risk neutral density

Breeden and Litzenberger (1978) show that a unique risk-neutral density f for a subsequent asset price S_T can be inferred from European call prices C(X) when contracts are priced for all strikes X and there are no arbitrage opportunities. The risk-neutral density (RND) is then given by

$$f(X) = e^{rT} \frac{\partial^2 C}{\partial X^2} \tag{1}$$

and the price of the call option is

$$C(X) = e^{-rT} \int_X^{\infty} (S - X) \cdot f(S) dS \tag{2}$$

[3]Some papers present the evolution of the parameters for different maturities. We are not able to follow the evolution of the parameters of the GB2 over time because this paper focuses only on one-month maturity options, due to liquidity restrictions.

where r stands for the risk-free interest rate and T the time to maturity. These relationships between the RND and derivative prices are the basis for empirical derivations of implied RND.

We employ a parametric approach to derive the RND. Assume that we have a parametric density function $f\left(X\left|\theta\right.\right)$ where is a parameter vector. Let $C_{market}(X_i)$ be the observed market price of call option at strike X_i. We obtain the RND by minimizing in θ the sum of squared difference between observed market prices and theoretical option prices

$$G(\theta) = \sum_{i=1}^{N} (C_{market}(X_i) - C(X_i\left|\theta\right.))^2 \tag{3}$$

with

$$C(X_i\left|\theta\right.) = e^{-rT} \int_{X_i}^{\infty} (x - X_i) \cdot f(x\left|\theta\right.)dx \tag{4}$$

where N is the number of prices obtained from option quotes or trades during a particular day for different strike prices X_i.

We use the generalized beta density of second kind (GB2) in equation (4). Bookstaber and McDonald (1987) presented the GB2 density. This distribution has four parameters $\theta = (a, b, p, q)$, allowing general combinations of the mean, variance, skewness and kurtosis of a variable. Hence, it is able to derive densities with flexible shape. The four parameters are positive, the parameter b is a scale parameter, and the product of the parameters a and q provides the maximum number of finite moments.

The GB2 density function is defined as

$$f_{GB2}\left(x\left|a,b,p,q\right.\right) = \frac{a}{b^{ap}B(p,q)} \frac{x^{ap-1}}{\left[1+(x/b)^a\right]^{p+q}}, x > 0 \tag{5}$$

with $B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$ and the Gamma function is $\Gamma(w) = \int_0^{\infty} e^{-u} \cdot u^{w-1}du$.

The density is risk-neutral when

$$F = \frac{b \cdot B(p + 1/a, q - 1/a)}{B(p,q)} \tag{6}$$

and its moments are

$$E[S_T^n] = \frac{b^n \cdot B\left(p + n/a, q - n/a\right)}{B(p,q)} \text{ for } n < aq \tag{7}$$

where the parameter b is determined by (6).

The theoretical option pricing formula depends on the cumulative distribution function (c.d.f.) of the GB2 density, denoted F_{GB2}, which is a function of the c.d.f. of the beta distribution, denoted F_β. We have

$$F_{GB2}(x \mid a, b, p, q) = F_{GB2}((x/b)^a \mid 1, 1, p, q) = F_\beta(h(x, a, b) \mid p, q) \tag{8}$$

with $h(x, a, b) = (x/b)^a \,/\, (1 + (x/b)^a)$.

If the density is risk-neutral, so that the constraint in equation (6) applies, then European call option prices are given by

$$C(X \mid \theta) = e^{-rT} \int_X^{\infty} (x - X) \cdot f_{GB2}(x \mid a, b, p, q) dx \tag{9}$$

$$\begin{aligned}
&= F \cdot e^{-r \cdot T} \left[1 - F_{GB2}\left(X \mid a, b, p + 1/a, q - 1/a\right)\right] \\
&- X \cdot e^{-r \cdot T} \left[1 - F_{GB2}\left(X \mid a, b, p, q\right)\right] \\
&= F \cdot e^{-r \cdot T} \left[1 - F_\beta\left(h(X, a\,,\, b) \mid p + 1/a, q - 1/a\right)\right] \\
&- X \cdot e^{-r \cdot T} \left[1 - F_\beta\left(h(X, a, b) \mid p, q\right)\right]
\end{aligned}$$

The parameter vector θ is estimated through the minimization of the option pricing error given by equation (3).

2.2 Evaluation of the performance of the forecasting ability of risk neutral densities

Let $\hat{F}(y_t)$ and $f\hat{\ }(y_t)$ denote the cumulative and probability density function forecasts made on day $t - 1$ for the exchange rate (y) on day t. Define the probability transform variable as

$$U(y_t) \equiv \int_{-\infty}^{y_t} f(u)\, du \equiv \hat{F}(y_t) \tag{10}$$

This variable captures the probability of obtaining a spot exchange rate lower than the realization, where the probability is calculated using the density forecast.[4] If the density forecast is correctly calibrated, then we should not be able to predict the probability of getting a value smaller than the realization. Therefore, a good density forecast implies that the transform variable is an independent and uniform variable on the $[0, 1]$ interval.

[4] Rosenblatt (1952).

Let $\Phi^{-1}(\cdot)$ be the inverse of the standard normal distribution function. Then we have the following result for any sequence of forecasts, regardless of the underlying distribution of portfolio returns. Berkowitz (2000) has shown that if the time series $x_t = \hat{F}(y_t) = \int_{-\infty}^{y_t} \hat{f}(u)du$ is distributed as an independent and identically distributed (iid) U(0,1), then $z_t = \Phi^{-1}\left(\int_{-\infty}^{y_t} \hat{f}(u)du\right)$ is an iid N(0,1).

Suppose we have generated the sequence $z_t = \Phi^{-1}(\hat{F}(y_t))$ for a given model. Since z_t should be independent across observations and standard normal, a wide variety of tests can be constructed. In particular, the null can for example be tested against a first-order autoregressive alternative with mean and variance possibly different than (0,1). We can write,

$$z_t - \mu = \rho\,(z_{t-1} - \mu) + \epsilon_t \tag{11}$$

where the null hypothesis $\mu = 0, \rho = 0$, and $var(\epsilon_t) = 1$.

A likelihood-ratio test of independence across observations can be formulated as,

$$LR_{ind} = -2 \cdot \left(L\left(\hat{\mu}, \hat{\sigma}^2, 0\right) - L\left(\hat{\mu}, \hat{\sigma}^2, \hat{\rho}\right)\right) \tag{12}$$

where the hats denote estimated values. This test statistic is a measure of the degree to which the data support a nonzero persistence parameter. Under the null hypothesis, the test statistic is distributed $\chi^2\,(\,1\,)$, chi-square with 1 degree of freedom, so that inference can be conducted in the usual way.

Of course, the null hypothesis is not just that the observations are independent but that they have mean and variance equal to (0,1). In order to jointly test these hypotheses, define the combined statistic as,

$$LR = -2 \cdot \left(L\,(0, 1\ , 0) - L\left(\hat{\mu}, \hat{\sigma}^2, \hat{\rho}\right)\right) \tag{13}$$

Under the null hypothesis, the test statistic is distributed $\chi^2\,(3)$. Since the LR test explicitly accounts for the mean, variance and autocorrelation of the transformed data, it should have power against very general alternatives.

3. Data Sampling and empirical results

In this study we use a set of prices of European call options written on Brazilian Real exchange rate (real/US dollar) from January 2000 to December 2005. Both options and futures prices were obtained from the Bolsa de Mercadoria de Futuros (BM&F). Due to liquidity restrictions we focus on 1-month maturity options.

The underlying asset in the Brazilian exchange rate options is the spot exchange rate. However, using closing prices for spot prices exchange rates may lead to problems with non-syncrhonicity. Spot closing prices at the end of the day are likely to be non-syncrhonous with the options markets. Therefore, in order to avoid such problems we employ exchange rate futures prices, which are also traded in BM&F, which have the same closing time.[5]

Our dataset contains 72 observations (months) and 216 options contracts. For each month at least three options (closest to the money) were selected. Therefore, we only consider the most liquid options to build our density forecasts.

Figure 1 presents the density forecast for January 2003. It is important to notice that the elections in 2002 were quite turbulent in the exchange rate market, due to market concerns regarding economic policy that would be implemented by the newly elected leftist party. Not only until mid 2003 these concerns proved wrong and market volatility was substantially reduced.

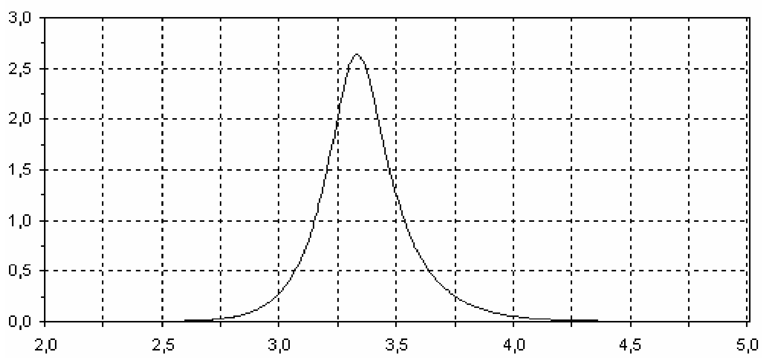

Figure 1
GB2 density forecast for January 2003

In order to test whether the density forecast is correctly calibrated, using the GB2 option pricing model, we model the z_t as a first order autoregressive process (AR(1)). The Berkowitz (2000) test yields a LR equal to 11.45, and the null hypothesis that the density provides a good forecast is reject at the 5% significance level. However, if we exclude an outlier from the analysis (April 2003) the LR reduces to 8.23 and we cannot reject the null hypothesis. In April 2003 the domestic exchange rate has had the highest one-month appreciation (12.5% against the US dollar). Therefore, it can be considered an outlier.

Figure 2 presents the density for December 2005, which is a more tranquil period for the exchange rate market (period of low volatility). As we can see the density is basically constrained in the [2,2.6] interval, which seems reasonable.

[5]We calculate spot prices using the methodology described in Andrade and Tabak (2001) and Chang and Tabak (2007).

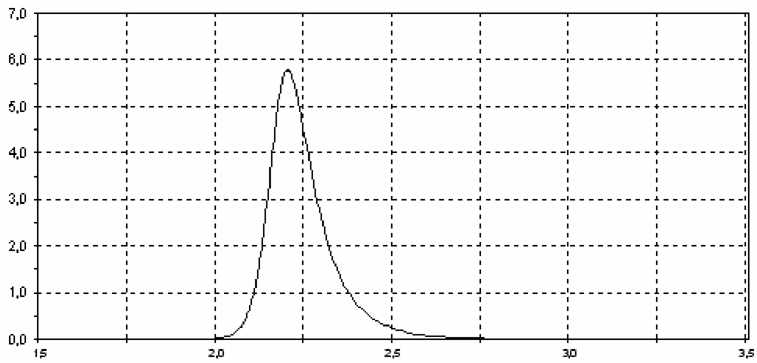

Figure 2
GB2 density forecast for December 2005

As a further check of robustness of this methodology we also evaluate the interval forecast for the exchange rate. We check whether realized exchange rate has fallen within the forecast interval, at the 95% confidence level, and computed the number of failures. Only 5 observations fall outside the 95% confidence interval in 72 observations, a failure rate of 6.94%. The Kupiec (1995) test is 1.29 with a p-value of 0.25, suggesting that the failure is close to the expected 5%. This result suggests that this methodology provides reasonable interval forecasts.

4. Conclusions

Density forecasting is essential for risk and portfolio management. Therefore, the development of models that are able to assess and provide good quality density forecasts has been in the research agenda for recent years.

This paper finds that density forecasts employing the generalized beta density of second kind (GB2) may be useful. This result is important as the exchange rate is one of the most important prices for many emerging markets and suggests that this methodology may add value in density forecasting. This study could be extended for different emerging markets and a variety of assets.

The empirical results suggests that allowing for flexible distributions, that incorporate skewness and kurtosis, yields satisfactory results, which imply that higher moments have to be considered in asset pricing models. This does not come as a surprise as it is well known that the traditional Black-Scholes model suffers from very restrictive assumptions on the dynamics of the underlying asset being priced.

Further research could compare different models that may be used to forecast asset price densities, and study more in depth the effects of incorporating skewness and kurtosis considerations into the analysis. Besides, asset price models have to be developed taking into account such considerations.

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