Nonparametric option pricing under Beta-t-GARCH process with dynamic conditional score

Manoel F. de S. Pereira†
Alvaro Veiga‡

Abstract One of the advantages of nonparametric option pricing methods is that they only require a set of future price scenarios, eliminating the need for an explicit risk-neutral model for the price of the underlying asset. In this paper, we explore the score-driven Beta-t-GARCH volatility model, introduced by Harvey (2013), to generate the price scenarios necessary for a nonparametric option pricing method based on the empirical Esscher transform, as proposed by Pereira and Veiga (2017). An experiment was conducted using real data from the Brazilian Stock Market, comparing observed option prices across different strike prices and maturities with the prices produced by two variants of the proposed method and those produced by parametric models, specifically Black and Scholes (1973) and Heston and Nandi (2000). The results indicate that the combined approach of the Beta-t-GARCH model and the empirical Esscher transform show significantly better outcomes most of the time.

Keywords: Nonparametric estimation; Dynamic conditional score; Option pricing; Empirical Esscher transform.
JEL Code: C1, C5, C6, G1.

1. Introduction

In their seminal paper, Black and Scholes (1973) derived a valuation formula for European options by modeling the prices of the underlying asset as a simple Gaussian Brownian Motion (GBM). Using a non-arbitrage argument, they obtained the risk-neutral process by altering the drift of the GBM. However, the Gaussian distribution struggles to explain several empirical stylized facts of financial returns, such as heavy tails, skewness, stochastic and mean-reverting volatility, among others (Tsay, 2013).

A vast body of literature has aimed to address this misspecification while still providing an explicit risk-neutral model for the underlying asset. Notable examples, based in the same non-arbitrage argument, include models by Cox and Ross (1976), Harrison and Kreps (1979), and Harrison and Pliska (1981).

Submitted on July 2, 2022. Revised on January 31, 2023. Accepted on January 31, 2023. Published online in September 2023. Editor in charge: Marcelo Fernandes.
†Universidade Federal Rural do Rio de Janeiro, Brazil: mpereira_tr@hotmail.com
‡Pontifícia Universidade Católica do Rio de Janeiro, Brazil: alvf@ele.puc-rio.br

A significant advance in addressing these misspecifications is integrating all empirical evidence into the pricing model. This means constructing a data-generating process that mirrors the stylized facts of the underlying asset while also using economic arguments and mathematical tools for risk neutralization. Within this parametric framework, the formulation of an explicit risk-neutral model for option pricing is limited to a handful of probability distributions.

For instance, the normality assumption fails to capture the negative skewness and excess kurtosis typical of log-returns. Resorting to stochastic jumps has become the favored approach to address the limitations of models with Gaussian innovations in continuous time(Bates, 2000; Eraker et al., 2003; Chernov et al., 2003).[1] In discrete time,[2] the results are pertinent only when the moment-generating function of the innovation distribution exists.[3] In such cases, we have multiple distributions to consider: Gaussian (Duan, 1995), Gamma (Siu et al., 2004), smoothly-truncated stable (Menn and Rachev, 2005), generalized error distributions (Christoffersen et al., 2006), and generalized hyperbolic (Chorro et al., 2008; Badescu et al., 2011).

When innovations exhibit heavy-tailed distributions, such as Student's innovations and historical asset returns, three methods can be applied, as shown by Liu et al. (2015). The first method states that real-world investors are risk-neutral (see Satoyoshi and Mitsui, 2011). The second method estimates parameters so that the underlying asset return implied by the model matches the risk-free interest rate (see Barone-Adesi et al., 2008). Both methods bypass an explicit change of measure. The third method, introduced by Badescu and Kulperger (2007), employs the extended Girsanov principle to derive the risk-neutral measure.

Other research has leveraged nonparametric estimation[4] to identify a risk-

[1] Christoffersen et al. (2011) argue that continuous-time models have become the cornerstone of modern option pricing theory. They provide closed-form solutions for European options and can incorporate stochastic volatility, leverage effects, and various types of risk premia (Heston, 1993; Bakshi et al., 1997; Broadie et al., 2007).

[2] Discrete-time settings, or GARCH frameworks, offer several advantages over continuous-time pricing models. They can serve as accurate numerical approximations of continuous-time models (Nelson and Cao, 1992; Nelson, 1996), eliminating discretization bias. Their predictions align perfectly with the filter used to extract variance (Harvey, 2013). Estimation is computationally efficient, volatility is observable at each time point, and they can incorporate multiple factors (Engle and Lee, 1999) and long memory (Bollerslev and Mikkelsen, 1996).

[3] Christoffersen et al. (2006), Heston (1993), and Heston and Nandi (2000) derived the risk-neutral measure from characteristic functions, but this is restricted to normal or inverse Gaussian innovations.

[4] There are two primary ways to nonparametrically estimate risk-neutral probabilities implicit in financial instruments: methods that infer the empirical risk-neutral probability from the options market and those that derive it from asset prices. Refer to Jackwerth (2004).

neutral measure for heavy-tailed distributions within GARCH models. Duan (2002) and Liu et al. (2015) utilized the canonical valuation,[5] as proposed by Stutzer (1996). In this approach, the measure change doesn't rely on the distribution of innovations, making it applicable even when the moment-generating function of the innovation's probability distribution is missing.

Several studies have expanded nonparametric option pricing in two main directions. The first direction demonstrates the methodology's flexibility and its accurate performance with realistic financial time series (see Gray and Newman, 2005; Alcock and Carmichael, 2008; Haley and Walker, 2010; Almeida and Azevedo, 2022). The second direction proposes other discrepancy functions as alternative measures of distance in the space of probabilities (see Haley and Walker, 2010; Almeida and Azevedo, 2022).

In this paper, we explore the versatility of nonparametric option pricing within a GARCH framework that incorporates non-Gaussian innovations. To represent a realistic financial time series, we employ a novel class volatility models, the dynamic conditional score, introduced by Harvey (2013).[6] These models define the parameters of the conditional distribution at time t as a linear function of parameters up to $t-1$ and the score function of the log-likelihood function. They are robust to extreme events and can capture the leverage effect by incorporating components that represent short and long-term volatilities. Harvey (2013) presents two models for volatility: Beta-t-(E)GARCH and Gamma-GED-(E)GARCH.[7] The focus here is on the Beta-t-GARCH model.[8]

In this study, we sidestep the formulation of a risk-neutral model. Instead, risk-neutralization is applied directly to the empirical distribution found in the sample paths generated from the assumed model, as seen in Liu et al. (2015) and Duan (2002). We obtain the empirical risk-neutral measure by applying

[5] The maximum entropy principle is employed to transform the empirical distribution of future sample asset returns into its risk-neutral counterpart by minimizing the Kullback-Leibler information criterion (KLIC).

[6] Harvey (2013) emphasizes that the model's goal is to capture heavy-tailed distributions with location μ_t and/or scale h_t evolving over time. The defining feature of these models is that their dynamics are propelled by the score of the conditional distribution. This allows for potential extensions to address other distributions with local linear trends, seasonality, skewed distributions, and time-varying skewness and kurtosis. Familiar models like GARCH can be expressed as specific instances of Score Driven Models.

[7] In the first, the conditional score is a linear function of a variable following a Beta distribution, while in the second, the conditional score is linearly related to a variable with a Gamma distribution.

[8] Harvey (2013) acknowledges that Beta-t-(E)GARCH models surpass Gamma-GED-(E)GARCH models. Furthermore, the Beta-t-GARCH model can be viewed as an approximation of the Beta-t-EGARCH model.

the empirical version of the Esscher transform (Esscher, 1932), called the empirical Esscher transform (EET), as proposed by Pereira and Veiga (2017), to the set of simulated price scenarios.

The primary contribution of this paper lies in the exploration of the Beta-t-GARCH models to generate the necessary set of price scenarios for the EET. We evaluate its pricing capability in relation to two established parametric benchmarks: Black and Scholes (1973) and Heston and Nandi (2000), using real data from call option prices on two stocks traded on the Brazilian Financial Market BOVESPA, specifically Vale and Petrobras.

The paper is structured as follows: section 2 introduces the proposed method. section 3 details the methodology used to compare the different pricing methods, and section 4 discusses the results. section 5 concludes the paper by summarizing the main findings.

2. Proposed method

First, we present the score-driven Beta-t-GARCH model by Harvey (2013), which will be used to generate a set of price scenarios for the underlying asset at maturity. Then, we describe the empirical Esscher transform (Pereira and Veiga, 2017), which uses the set of price scenarios as input to produce an empirical risk-neutral distribution and, ultimately, the fair value of the option, based on a non-arbitrage argument.

2.1 Score-driven Beta-t-GARCH model for financial returns

Let y_t be the log-return of the underlying asset at time t, defined by

$$y_t = \ln\left(S_t/S_{t-1}\right), \quad t = 1,\dots,T, \tag{1}$$

where S_t is the asset price at time t. Let $f(y_t|\widetilde{Y}_{t-1},\theta_t)$ be the density of y_t conditioned on its past values, in $\widetilde{Y}_{t-1} \equiv \{y_1,y_2,\dots,y_{t-1}\}$, with parameter vector θ_t. In a score general driven model, the evolution of the vector of parameters θ_t is described by the dynamic equation

$$\theta_{t+1} = \kappa + As_t + B\theta_t, \tag{2}$$

where κ, A, and B are fixed vectors and matrices with appropriate dimensions and $s_t = \psi(\theta_t)\nabla(y_t,\theta_t)$, with $\psi(\theta_t)$ a positive definite scaling matrix and $\nabla(y_t,\theta_t) = \partial \ln f(y_t|\widetilde{Y}_{t-1},\theta_t)/\partial\theta_t$ the *score* of $f(y_t|\widetilde{Y}_{t-1},\theta_t)$.

The scaling matrix $\psi(\theta_t)$ is defined in several different ways in the literature. Here we follow Creal et al. (2013) and suggest defining it as a power p

of the inverse of the information matrix, i.e.

$$\psi(\theta_t) = E\left[\frac{\nabla(y_t,\theta_t)\nabla^T(y_t,\theta_t)}{\widetilde{Y}_{t-1}}\right]^{-p}, \quad p \in \mathrm{Z}^+. \tag{3}$$

Different models are automatically obtained by specifying the conditional density $f(y_t|\widetilde{Y}_{t-1},\theta_t)$. In many cases, well-known models are obtained. For example, if we define $f(\cdot)$ as a Gaussian distribution with zero mean and $\theta_t = V[y_t]$ we obtain the usual GARCH model (Bollerslev and Mikkelsen, 1996).

Now, write the log-return y_t as a fixed mean process with conditional heteroskedasticity and a standardized error term z_t proportional to a t-student random variable ε_t, i.e

$$y_t = \mu + \sqrt{h_t} z_t, \tag{4}$$

where $z_t = \sqrt{(\upsilon-2)/\upsilon}\,\varepsilon_t$, $\upsilon > 2$, and $\varepsilon_t \sim t_\upsilon$. Substituting $\theta_t = [\mu, h_{t-1}, \upsilon]^T$ into equation (2), the conditional density of y_t can be written as

$$f\left(y_t \middle| \widetilde{Y}_{t-1},\mu,h_t,\upsilon\right) = \frac{\Gamma((\upsilon+1)/2)}{\Gamma(\upsilon/2)\sqrt{h_t\pi(\upsilon-2)}}\left(1+\frac{(y_t-\mu)^2}{(\nu-2)h_t}\right)^{-\frac{\upsilon+1}{2}}, \tag{5}$$

where $\upsilon > 2$.

Then, we verify that the score of $f(y_t|\widetilde{Y}_{t-1},\mu,h_{t-1},\upsilon)$ can be written as

$$\nabla(y_t,\mu,h_t,\upsilon) = -\frac{1}{2h_{t-1}}\left\{1-(\upsilon+1)\left[\frac{\frac{m^2}{\omega}}{1+\frac{m^2}{\omega}}\right]\right\} \tag{6}$$

with $m = (y_t - \mu)$ and $\omega = (\upsilon-2)h_t$. Harvey (2013) shows that the variable $(m^2/\omega)/(1+m^2/\omega)$ follows a $\mathrm{Beta}(1/2,\upsilon/2)$ distribution. Consequently, $\nabla(y_t,h_t)$ has, as expected, zero mean and finite variance.

Setting $p = 1$ in (3), we obtain

$$\psi(\mu,h_t,\upsilon) = 2h_t^2 \tag{7}$$

which leads to

$$s_{t-1} = \psi(\mu,h_{t-1},\upsilon)\nabla(y_{t-1},\mu,h_{t-1},\upsilon) = h_{t-1}r_{t-1} \tag{8}$$

with

$$r_{t-1} = \left\{(\upsilon+1)\left[\frac{\frac{m^2}{\omega}}{1+\frac{m^2}{\omega}}\right]-1\right\}, \; -1 \leqslant r_{t-1} \leqslant \upsilon. \tag{9}$$

Note that r_t has a known distribution with zero mean since it is a linear function of a Beta$(1/2, \upsilon/2)$ random variable. We can then express the score-driven dynamic equation for the stochastic variance h_t of the Beta-t-GARCH(1,1) model as

$$h_t = \delta + (\phi + \alpha r_{t-1}) h_{t-1}. \tag{10}$$

The sufficient conditions for the conditional variance to remain positive are $\delta > 0$, $\phi \geqslant 0$, $\alpha \geqslant 0$, and $\phi - \alpha \geqslant 0$. For this last condition, note that $r_t \in [-1, \upsilon]$.

Given these conditions, $h_t > 0$, and the drift criterion of Meyn and Tweedie (1994) applies directly, showing that $0 \leqslant \phi < 1$ is a sufficient condition for the process to be strictly stationary and ergodic. This implies that h_t converges in distribution, which ensures the consistency of the fixed parameters of the model.

In our exercises, an extended version of the model is used, as suggested by Harvey (2013), to include leverage effects by adding the indicator variable $I(y_{t-1} < 0)$, as follows:

$$h_t = \delta + \phi h_{t-1} + \alpha h_{t-1} r_{t-1} + I(y_{t-1} < 0)\, \alpha^* h_{t-1} r_{t-1}. \tag{11}$$

The conditions for the positivity of h_t are analogous to the previous case: $\delta > 0$, $\phi \geqslant 0$, $\alpha \geqslant 0$, $\alpha^* \geqslant 0$, and $\phi \geqslant \max(\alpha, \alpha^*)$.

In our method, the pricing of an option at time t^* is carried out in two steps. First, we estimate the fixed parameters δ, ϕ, α, α^*, μ, and υ using data up to t^*. Then, we compute the sequence of $h_t^{'}s$ up to time t^*. Finally, we simulate trajectories for $y_{t^*+1}, \ldots, y_{t^*+\tau}$ with τ being the maturity of interest using equations (4) and (11).

2.2 Option pricing via Empirical Esscher transform

The Esscher Transform (ET – Esscher, 1932) of a probability density function (pdf) $f(z)$ is defined as a function of a parameter λ, termed the Esscher parameter, and is given by

$$f(z;\lambda) = \frac{e^{\lambda z}}{\int_{-\infty}^{+\infty} e^{\lambda w} f(w) dw} f(z). \tag{12}$$

The ET $f(z;\lambda)$ is a distorted version of $f(z)$ and is also a pdf since it integrates to one. For instance, if $f(z)$ is a Gaussian distribution, $f(z;\lambda)$ will also be Gaussian with a different mean, provided $\lambda \neq 0$.

Gerber and Shiu (1994) proposed using the ET to determine the so-called 'risk-neutral measure' Q. According to the fundamental theorem of asset pricing (Binghan and Kiesel, 2004), the value of a derivative is simply the expected value of the payoff discounted by the risk-free rate of return.

Let $g(S_T)$ represent a derivative and S_T be the price of the underlying at its maturity, at time T. The risk-neutral value $v(g(S_T)/S_0)$ of the derivative, measured at time $t=0$, is given by

$$v\left(\frac{g(S_T)}{S_0}\right) = \mathrm{e}^{-rT} E^Q\left[\frac{g(S_T)}{S_0}\right].$$

Let $Y_T = \sum_{t=1}^{T} y_t$ denote the cumulative log-return for T time periods. Given that $S_T = S_0 \mathrm{e}^{Y_T}$, $f(Y_T|\mathscr{F}_0)$ is its pdf conditioned on the information set at $t=0$, $\mathscr{F}_0$, and $f(Y_T;\lambda|\mathscr{F}_0)$ is the corresponding Esscher Transform. Then,

$$v\left(\frac{g(S_T)}{S_0}\right) = \mathrm{e}^{-rT}\int_{-\infty}^{+\infty} g\left(S_0\mathrm{e}^{Y_T}\right) f(Y_T;\lambda|\mathscr{F}_0)\,\mathrm{d}Y_T. \tag{13}$$

This holds true even if the derivative is the asset itself, implying $g(S_T) = S_T$ and $v(g(S_T)/S_0) = S_0$. This establishes the non-arbitrage constraint,

$$\underbrace{\mathrm{e}^{-rT}\int_{-\infty}^{+\infty} S_0\mathrm{e}^{Y_T} f(Y_T;\lambda|\mathscr{F}_0)\,\mathrm{d}Y_T}_{=S_0} \to \mathrm{e}^{rT} = \int_{-\infty}^{+\infty} \mathrm{e}^{Y_T} f(Y_T;\lambda|\mathscr{F}_0)\,\mathrm{d}Y_T. \tag{14}$$

Expressing the risk-neutral measure $f(Y_T;\lambda|\mathscr{F}_0)$ as the ET of $f(Y_T|\mathscr{F}_0)$ yields

$$\mathrm{e}^{rT} = \frac{\int_{-\infty}^{+\infty}\mathrm{e}^{(\lambda+1)Y_T} f(Y_T|\mathscr{F}_0)\,\mathrm{d}Y_T}{\int_{-\infty}^{+\infty}\mathrm{e}^{\lambda Y_T} f(Y_T|\mathscr{F}_0)\,\mathrm{d}Y_T} \to \mathrm{e}^{rT} = \frac{M(\lambda+1)}{M(\lambda)}, \tag{15}$$

where $M(\lambda)$ is the moment generating function (mgf) of $Y_T|\mathscr{F}_0$.

Following the Gerber and Shiu (1994) methodology for derivative pricing, one must assume a distribution for $Y_T|\mathscr{F}_0$, compute its mgf, solve (15),

$$\lambda^* = \arg_\lambda\left\{\mathrm{e}^{rT} = \frac{M(\lambda+1)}{M(\lambda)}\right\},$$

then compute the ET using (12), and finally, use (13) to determine the risk-neutral value of the derivative.

The Gerber and Shiu (1994) methodology is limited to cases where $f(Y_T|\mathscr{F}_0)$ is known. This poses a significant constraint since, even for the simple case where log-returns follow the widely recognized GARCH process (Bollerslev, 1987), $f(Y_T|\mathscr{F}_0)$ is unknown for $T>1$. A potential solution to this limitation is the Empirical Esscher transform (EET) proposed by Pereira and Veiga (2017). Consider a random sample of cumulative log-returns of size n from $f(Y_T|\mathscr{F}_0)$, denoted by $\{Y_{T,i}\}_{i=1}^n$. The EET is defined as

$$q_{i,\lambda}=\frac{\mathrm{e}^{\lambda Y_{T,i}}}{\sum_{j=1}^n \mathrm{e}^{\lambda Y_{T,j}}}.$$

Observe that $\left\{q_{i,\theta}\right\}_{i=1}^n$ forms a probability mass function on its own since $\sum_{i=1}^n q_{i,\theta}=1$ and $q_{i,\theta}>0$, $\forall i$. It's worth noting that $\left\{q_{i,\theta}\right\}_{i=1}^n$ can be viewed as a reweighted version of the original sample $\{Y_{T,i}\}_{i=1}^n$:

$$q_{i,\lambda}=m_q\left(Y_{T,i};\lambda\right)p_i,$$

where $p_i=1/n$ is the original weight and $m_q\left(Y_{T,i};\lambda\right)$ is the reweighting function, given by

$$m_q\left(Y_{T,i};\lambda\right)=\frac{\mathrm{e}^{\lambda Y_{T,i}}}{\frac{1}{n}\sum_{j=1}^n \mathrm{e}^{\lambda Y_{T,j}}}.$$

With the EET, the non-arbitrage condition (14) can be reformulated as

$$\mathrm{e}^{rT}=\sum_{i=1}^n \mathrm{e}^{Y_{T,i}}q_{i,\lambda}=\frac{\sum_{i=1}^n \mathrm{e}^{(\lambda+1)Y_{T,i}}}{\sum_{j=1}^n \mathrm{e}^{\lambda Y_{T,j}}}\quad \text{or}\quad \mathrm{e}^{rT}=\frac{\widehat{M}(\lambda+1)}{\widehat{M}(\lambda)},$$

where

$$\widehat{M}(\lambda)=\frac{1}{n}\sum_{j=1}^n \mathrm{e}^{\lambda Y_{T,j}}.$$

The empirical Esscher parameter is then expressed as

$$\widehat{\lambda}^*=\arg_\lambda\left\{\mathrm{e}^{rT}=\frac{\widehat{M}(\lambda+1)}{\widehat{M}(\lambda)}\right\}. \tag{16}$$

The weak law of large numbers ensures that if $E\left[\mathrm{e}^{\lambda y_T}\middle|\mathscr{F}_0\right]$ and $E\left[2\mathrm{e}^{\lambda y_T}\middle|\mathscr{F}_0\right]$ exist for all $\lambda\in\mathfrak{R}$, then $\widehat{M}(\lambda)$ is a consistent estimator of $M(\lambda)$, i.e.,

$$\widehat{M}(\lambda)\xrightarrow{P}M(\lambda).$$

Therefore, by invoking the continuous mapping theorem by Mann and Wald (1943), the empirical Esscher parameter will also converge to the Esscher Parameter, i.e.,

$$\widehat{\lambda}^*\xrightarrow{P}\lambda^*.$$

It is noteworthy that the same risk-neutral measure is achieved by the canonical valuation proposed by Stutzer (1996) using a distinct approach. However, his framework does not accommodate the convergence results offered by the EET.

3. Methodology

This section presents the methodology used to compare our proposal with the Black and Scholes (1973) and Heston and Nandi (2000) models in experiments using recent data.

3.1 Data

Two data sets have been used in this analysis. The first set comprises 260 daily closing prices for Vale and Petrobras from January 2nd, 2011, to January 17th, 2012. This set will be used to estimate both the Black-Scholes and Heston-Nandi models, and to simulate price trajectories through bootstrapping. The second data set consists of option market prices for Vale and Petrobras with a range of different maturities and strike prices on January 18th, 2012, the day following the period covered by the first data set. In our exercise, we are at $t = 0$, which is the moment after the market closed on January 17th, 2012. The closing prices for that day are $S_0 =$ R$ 41.13 for Vale and $S_0 =$ R$ 24.37 for Petrobras.

Table 1 presents the main descriptive statistics of the log returns of Vale and Petrobras. Both show a negative mean, similar risk levels, negative skewness, and excess kurtosis. Moreover, the Ljung-Box test indicates that the squared log returns are serially correlated, while the simple log returns are not for both series. This, combined with the excess kurtosis, indicates the necessity of using volatility models.

Figures 1 and 2 show the price behavior during the studied period. Since the 2008 crisis, the prices of these companies have followed a downward trend. The primary factors contributing to the decline in prices were the Arab Spring, the downgrade of the US credit rating by Standard and Poor's, and the crisis in the Eurozone (Greece, Italy, Ireland, and Portugal declared an inability to pay their debts).

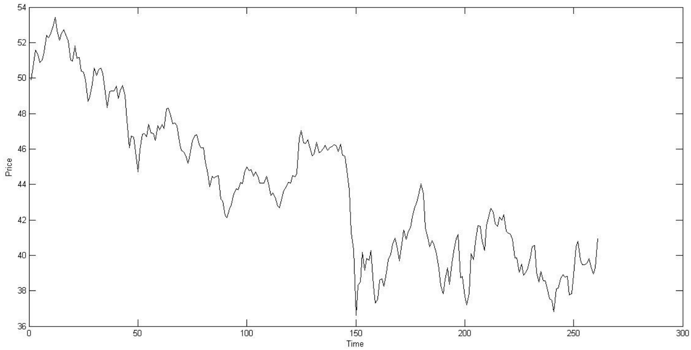
Table 2 displays the market closing prices of options for Vale and Petrobras for different strikes and maturities: 17/252, 40/252, 59/252, and 121/252 years.[9] The risk-free interest rate is derived from the forward curve of swap

[9] All required data were obtained from B3 (http://www.b3.com.br). Accessed on 12/20/2017.

Table 1
Descriptive statistics of the log returns

	Mean	Std. Dev.	Skewness	Kurtosis	Maximum	Minimum
Petrobras	0.00	0.02	–0.56	5.22	0.05	–0.08
Vale	0.00	0.02	–0.65	7.14	0.06	–0.10

Figure 1
Vale prices from January 17th, 2011, to January 17, 2012



Contracts DI X PRE,[10] with maturities of 30, 60, 90, and 120 days.[11] The final values are 10.3499% for maturity 17/252, 10.2485% for maturity 40/252, 10.1721% for maturity 59/252, and 10.032% for maturity 121/252, determined by linear interpolations.[12]

3.2 Pricing models and methods

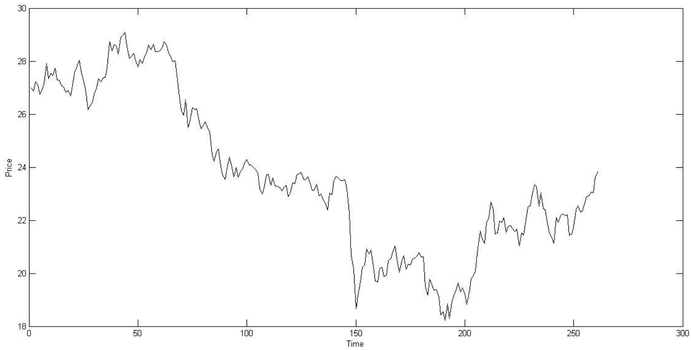
In our comparison exercise, we evaluate three pricing methodologies: the parametric Black-Scholes and Heston-Nandi models, and our non-parametric pricing methodology (EET) using scenarios generated by Harvey's Score-Driven Beta-t-GARCH model. Additionally, we also consider EET using sce-

[10] The DI spot rate is the overnight interbank deposit rate, representing the interest rate at which a depository institution lends immediately available funds to another depository institution. It offers an efficient method for banks to access short-term financing from central bank depositories.
[11] `http://www.bcb.gov.br`. Accessed on 12/20/2017.
[12] According to Hagan and West (2006), the primary issues with interpolation algorithms are that they may not be arbitrage-free and/or they might lead one to derive unreasonable hedging strategies.

Table 2
Market prices of options in Vale and Petrobras

	Vale (VALE5)		Petrobras (PETR4)	
Maturity (days)	Strike	Option price	Strike	Option price
17	44.00	0.11	25.66	0.18
	43.14	0.21	25.16	0.27
	42.00	0.54	24.83	0.43
	41.14	0.99	23.66	1.12
	41.00	1.03	22.83	1.80
	40.14	1.72	21.66	2.88
	39.07	2.50	20.83	3.63
	38.57	2.75	19.66	4.94
	37.14	4.24	18.66	5.78
	37.00	4.41	17.66	6.78
	36.14	5.32	15.16	9.19
	36.00	5.34		
	35.00	6.05		
	34.00	7.31		
	30.14	11.41		
	30.00	11.65		
	28.00	13.59		
40	46.07	0.15	27.83	0.09
	46.00	0.20	27.00	0.17
	45.57	0.19	25.83	0.40
	44.07	0.42	25.33	0.61
	43.07	0.86	25.00	0.77
	42.07	1.25	23.83	1.38
	42.00	1.32	22.83	2.11
	41.00	1.83	21.66	3.08
	40.57	2.01	21.00	3.70
	40.00	2.51	19.66	4.95
	38.00	4.00	18.66	5.79
	37.00	4.75	17.83	6.60
	36.57	5.21		
	35.07	6.50		
	32.00	9.30		
59	48.00	0.21	26.00	0.58
	44.14	0.87	24.00	1.50
	44.00	0.90	21.83	3.21
	41.07	2.30		
	40.00	3.00		
121			25.50	1.85

Figure 2
Petrobras prices from January 17th, 2011, to January 17, 2012



narios generated by bootstrapping the historical log-returns from the first data set.

3.2.1 Proposed pricing method: EET + Beta-t-GARCH model

Our pricing methodology consists in four steps:

Step 1. Consider one year (252 days) of daily prices of the underlying and estimate the Beta-t-GARCH model.
Step 2. With the estimated model, generate a sample of size n of the underlying prices at maturity, $S_{T,i}$, where $i = 1,\ldots,n$.
Step 3. Evaluate the EET for this sample and compute the parameter, $\widehat{\lambda}^*$.
Step 4. With the transformed sample, compute the option price, given by the mean value of the payoff at maturity discounted by the risk-free rate of interest.

Once the model is estimated in step 1, we generate a sample of size 252 of possible cumulated returns $\{Y_{T,i}\}_{i=1}^{252}$ using model (1) to (11) for each maturity $T = 17/252$, $40/252$, $59/252$, and $121/252$. In step 3, we compute the empirical Esscher parameter, $\widehat{\theta}^*$, using the equation (16), as below:

$$\widehat{\lambda}^* = \arg_\theta \left\{ e^{rT} = \frac{e^{\lambda Y_{T,i}}}{\sum_{j=1}^{n} e^{\lambda Y_{T,j}}} \right\},$$

and, finally, in step 4, we calculate the option price

$$C(K,T) = \mathrm{e}^{-rT}\left[\sum_{j=1}^{n}(S_{T,i}-K)^{+}\frac{\mathrm{e}^{\widehat{\theta}^{*}Y_{T,i}}}{\sum_{j=1}^{n}\mathrm{e}^{\widehat{\theta}^{*}Y_{T,j}}}\right], S_{T,i} = S_0\mathrm{e}^{Y_{T,i}}.$$

We refer to this result as the *price EET-BtG* (Empirical Esscher Transform with Beta-t-GARCH Model). Since the option price estimate obtained from this methodology is subject to sample variation, we repeat the procedure 15,000 times and calculate the mean price. We then calculate the Absolute Percentage Error (APE) in relation to the option market price.

We also tried a different combination with a sample size of 50,000 with 200 repetitions to analyze if the accuracy increases with the sample size. We calculate the mean price and refer to this variant as *price EET-BtG**. We then calculate the APE in relation to the option market price.

3.2.2 Variation of proposed pricing method: EET + Bootstrapping

In this variation, the Beta-t-GARCH model is replaced when simulating trajectories for the log-return with a bootstrapping procedure, as in Pereira and Veiga (2017). The modified algorithm is then:

1. Take a bootstrap sample of size T from the historical log-returns in the first database.
2. Compute the cumulated log-return $Y_{T,i}$.
3. Repeat steps (1) and (2) with $i = 1,\ldots,252$ to produce the sample $\{Y_{T,i}\}_{i=1}^{252}$.
4. Evaluate the EET for this sample and compute the parameter, $\widehat{\lambda}^{*}$.

This is the *price EET-B* (Empirical Esscher Transform with bootstrap). As before, we repeat this procedure 15,000 times and calculate the mean price. We then calculate the APE in relation to the option market price. As before, we repeat the procedure with a sample size of 50,000 with 200 repetitions and refer to this variant as *price EET-B**. We then calculate the APE in relation to the option market price.

3.2.3 The Black-Scholes model

The Black and Scholes (1973) formula for the price of options assumes that stock prices follow a geometric Brownian motion with normally distributed logarithmic returns and constant drift and volatility. The pricing formula for a European call option with maturity T on a non-dividend-paying stock is given by

$$C = S_0 N(d_1) - K\mathrm{e}^{-rT} N(d_2),$$

where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

The functions $N(d_1)$ and $N(d_2)$ represent the cumulative probability distribution functions for a normal variable with zero mean and variance equal to 1. C is the price of the call option, S_0 is the stock price at time 0, r is the risk-free interest rate continuously compounded, and σ is the asset volatility. In Black-Scholes prices, we use the annualized historical volatility. We refer to this as the *price BSM* (Black-Scholes Model). We then calculate the APE in relation to the option market price.

3.2.4 The Heston-Nandi model

The Heston and Nandi (2000) model assumes that the log-returns $y_t = \ln(S_t) - \ln(S_{t-1})$ follow a GARCH (1,1) in-the-mean process driven by the following pair of equations, under the physical measure:

$$y_t = r + \lambda\sigma_t^2 + \sigma_t z_t,$$

with

$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha\left(z_{t-1} - \gamma\sigma_{t-1}\right)^2,$$

where r is the risk-free interest rate, λ represents the risk premium, σ_t^2 is the conditional variance, and z_t is the error term distributed as a standard normal variable, $z_t \sim N(0,1)$. The parameter α determines the degree of kurtosis, γ determines the skewness, and variance persistence is given by $\beta + \alpha\gamma^2$. The process will be mean-reverting if $\beta + \alpha\gamma^2 < 1$. The risk-neutral version of this model can be written as

$$y_t = r - \frac{1}{2}\sigma_t^2 + \sigma_t z_t^*$$

$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha\left(z_{t-1}^* - \gamma^*\sigma_{t-1}\right)^2,$$

where λ is replaced by $-1/2$, z_t^* is defined as $z_t^* = z_t + \left(\lambda + \frac{1}{2}\right)\sigma_t$, and γ is replaced by $\gamma^* = \gamma + \lambda - 1/2$. The price at time t of a European call option with maturity at time $t+T$ is given by:

$$C = \mathrm{e}^{-rT} E_t^* \left[(S_{t+T} - K)^+ \right] = S_t P_1 - K \mathrm{e}^{-rT} P_2,$$

where T is the time to maturity, $E_t^*[S_t]$ is the expectation of S_t under the risk-neutral distribution, S_t is the price of the underlying asset at time t, and P_1 and P_2 are the risk-neutral probabilities. We refer to this as the *price HN* (Heston-Nandi). We then calculate the APE regarding the option market price.

The quantities P_1 and P_2 can be obtained by inverting the characteristic functions $f^*(\phi)$:

$$P_1 = \frac{1}{2} + \frac{\mathrm{e}^{-rT}}{\pi S_t} \int_0^\infty \mathrm{Re} \left[\frac{K^{-i\phi} f^*(i\phi + 1)}{i\phi} \right] \mathrm{d}\phi$$

$$P_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re} \left[\frac{K^{-i\phi} f^*(i\phi)}{i\phi} \right] \mathrm{d}\phi$$

4. Results

Before beginning the estimation procedure, we confirmed that the Ljung-Box test fails to reject the "zero autocorrelation hypothesis" for log-returns and does reject it for the square log-returns in both series. All parameters were estimated using the maximum likelihood method from the first data set, which contains 260 daily closing prices for Vale and Petrobras.[13]

Table 3 displays the results for the Heston-Nandi model. The sufficient condition for the conditional variance to remain positive was met ($\widehat{\omega} > 0$, $\widehat{\beta} \geqslant 0$, $\widehat{\alpha} \geqslant 0$, $\widehat{\gamma} \geqslant 0$, and $\widehat{\lambda} \geqslant 0$), and the process is mean-reverting because $\widehat{\beta} + \widehat{\alpha}\widehat{\gamma}^2 < 1$.

Table 4 provides estimates for the Beta-t-GARCH model, with the associated standard errors in parentheses. All estimates are significant at the 5% level and meet the sufficient conditions for the conditional variance to be positive, i.e., $\widehat{\delta} > 0$, $\widehat{\phi} \geqslant 0$, $\widehat{\alpha} \geqslant 0$, and $\widehat{\alpha}^* \geqslant 0$. Moreover, y_t is strictly stationary and ergodic since $\phi < 1$. The estimated degrees of freedom (υ) suggest a distribution with heavy tails, as anticipated. The leverage effect parameter $(\widehat{\alpha}^*)$ also indicates that a positive y_t contributes $\alpha u_{t-1} h_{t-1}$ to h_t, while a negative

[13] Regarding parameter optimization, we employed the following heuristics: We began with a random initial condition and utilized the Nelder-Mead optimization methodology. The values of the resulting parameters served as an initial solution for the BFGS optimization method. Subsequently, the Nelder-Mead method was applied again, and this process continued until the difference between successive solutions was less than a tolerance value of 0.10.

Table 3
Estimated parameters of Heston-Nandi model

	Petrobras				Vale		
	T = 17/252	**T = 40/252**	**T = 59/252**	**T = 121/252**	**T = 17/252**	**T = 40/252**	**T = 59/252**
$\widehat{\omega}$	8.24×10^{-7}	2.77×10^{-6}	2.27×10^{-6}	3.29×10^{-8}	4.07×10^{-5}	2.86×10^{-5}	1.68×10^{-5}
$\widehat{\beta}$	9.91×10^{-1}	9.82×10^{-1}	9.83×10^{-1}	9.88×10^{-1}	8.43×10^{-1}	8.81×10^{-1}	9.30×10^{-1}
$\widehat{\alpha}$	1.98×10^{-6}	3.02×10^{-6}	3.17×10^{-6}	3.75×10^{-6}	3.24×10^{-6}	4.09×10^{-6}	2.96×10^{-6}
$\widehat{\gamma}$	1.99×10^{1}	3.57×10^{-6}	6.48×10^{-6}	4.85×10^{-8}	6.79	1.61×10^{-6}	5.18×10^{-5}
$\widehat{\lambda}$	3.04×10^{-12}	8.80×10^{-2}	6.64×10^{-2}	1.00×10^{-1}	1.07×10^{-10}	8.52×10^{-5}	2.28×10^{-7}

Table 4
Estimated parameters of Beta-t-GARCH model

	$\hat{\delta}$	$\hat{\phi}$	$\hat{\alpha}$	$\hat{\alpha}^*$	$\hat{\mu}$	$\hat{\nu}$
Petrobras	2.09×10^{-5}	9.31×10^{-1}	5.77×10^{-2}	6.84×10^{-2}	-1.31×10^{-4}	6.99
	(1.98×10^{-6})	(6.72×10^{-3})	(4.43×10^{-3})	(5.92×10^{-3})	(6.05×10^{-6})	(1.82×10^{-1})
Vale	2.76×10^{-5}	9.01×10^{-1}	1.63×10^{-1}	4.13×10^{-2}	-6.42×10^{-4}	6.40
	(9.93×10^{-7})	(3.96×10^{-3})	(6.02×10^{-3})	(7.86×10^{-3})	(5.23×10^{-5})	(1.45×10^{-1})

y_t has a more substantial impact $(\widehat{\alpha}+\widehat{\alpha}^*)u_{t-1}h_{t-1}$ with $\widehat{\alpha}^* \geqslant 0$. The estimated conditional mean is negative, as shown in Table 1, but is lower than the corresponding conditional mean of raw log-returns.

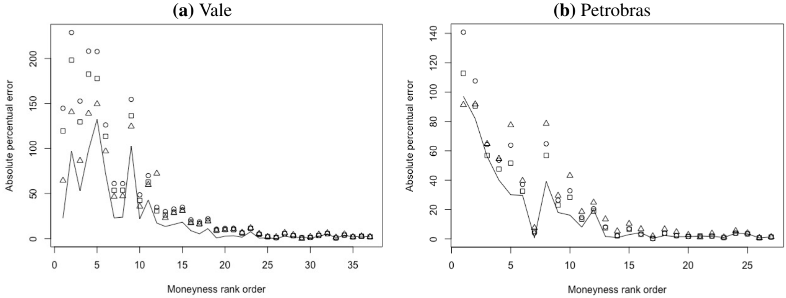
Standardized residuals and their squares showed no serial correlation according to the Ljung-Box test for both models, suggesting their adequacy. Moreover, the null of the Jarque-Bera test was strongly rejected, indicating that the models have been correctly fitted.

Tables 5 and 6 present the APEs obtained by the EET-BtG, EET-BB, BS, and HN models for options on Petrobras and Vale, respectively. Errors are calculated regarding the closing market prices of the option for a specific day, using one year of past data up to the market closing of the preceding day.

Figures 3a and 3b display the pricing errors ordered by moneyness for Petrobras and Vale, respectively. For clarity, we chose not to display the actual scale of moneyness. The black bottom line represents the error obtained by our proposed EET-BtG model, while the dots represent the error for EET-B, BS, and HN.

The pricing errors for all methods decrease significantly as moneyness increases. This trend is because pricing out-of-the-money options is challenging, as previously highlighted by Gray and Newman (2005) and Haley and Walker (2010). Notably, our proposed EET-BtG model consistently exhibits lower pricing errors compared to other methods for both Petrobras and Vale

Figure 3
Absolute Percentage Error for VALE (3a) and Petrobras (3b)



Black line: EET-BtG, square: EET-B, circle: BS, triangle: HN. For clarity, moneyness is not represented in its original scale but just by its rank order. Low level errors (around 3% or less) are observed for moneyness greater than 1.05.

options. For Petrobras options, EET-BtG had the lowest MAPE in 23 out of 27 cases, and for Vale options, it had the lowest MAPE in 34 out of 37 cases. In the remaining cases, EET-AM was very close to the top-performing model.

For Petrobras, the second-best model was EET-B, followed closely by BS, with HN surprisingly underperforming. However, for Vale, HN was distinctly the second-best model, outperforming EET-B and BS in 27 out of 37 cases.

We attribute the comparatively positive results for EET-BtG to two primary factors. First, Pereira and Veiga (2017) demonstrated through a simulation study that EET can closely mimic the pricing formulas of BS and HN. In our experiment, if the data adhered to one of these models, EET-BtG should accurately replicate their pricing. Additionally, the BtG model, unlike BS and HN, is tailored for robustness against the extreme market fluctuations observed in historical prices for the specified timeframe.

On the other hand, two factors might contribute to pricing errors: synchronization mismatch and, more significantly, liquidity risk. The synchronization mismatch occurs because the closing price of the underlying asset used in this study differs from the closing price of the options at the time of trade. This discrepancy can lead to substantial pricing errors (George and Longstaff, 1993) and spurious arbitrage opportunities (Galai, 1979). The liquidity risk associated with the derivative significantly influences market prices, causing notable deviations from the "fair price" determined by models (Pérignon and Villa, 2002). None of the models or methods in our analysis can explain this effect.

Tables 7 and 8 display the results of the comparison between the relative errors for the proposed method for different sample sizes for the Petrobras and Vale databases, respectively. We increased the sample size of the empirical distribution to analyze its impact on prices calculated by the proposed method (EET-AM* and EET-B*). Generally, the results do not show a significant reduction in pricing errors that would justify the high computational cost of generating scenarios.

5. Conclusions

In this study, we propose and evaluate a new option pricing method termed *EET-BtG*. This method is based in the Empirical Esscher Transform as presented by Pereira and Veiga (2017), and employs price scenarios generated by the dynamic conditional score model Beta-t-GARCH, as proposed by Harvey (2013). Notably, this model is robust to outliers and makes the modeling of the leverage effect easier, incorporating both short-term and long-term volatility components.

A primary advantage of our method, similar to other non-parametric techniques, is its ability to bypass the necessity of constructing an explicit risk-neutral model for returns. Thus, parametric pricing becomes intricate since it requires the alignment of both physical and risk-free models using compatible parameters.

With our approach, the physical model is decoupled from the pricing procedure. A sample of prices for the underlying asset is produced under the physical model, which is subsequently reweighted using EET, yielding a risk-neutralized sample. The option's price is then straightforwardly determined by the weighted mean of the pay-offs within this sample, discounted at the risk-free interest rate.

We unite our method with two established competing models: Black and Scholes (1973) and Heston and Nandi (2000). We assess the pricing capabilities of these models for options across various strike prices and maturities, focusing on two assets traded in the Brazilian financial market: Vale and Petrobras.

Our proposed method, EET-BtG, showcased the APEs across all maturities and moneyness levels. Broadly speaking, our findings align with existing literature. It was possible to observe that pricing errors decrease as moneyness escalates (refer to Gray and Newman, 2005; Haley and Walker, 2010) and are more pronounced for deep-out-of-the-money and out-of-the-money options (Gencay and Salih, 2003).

The promising outcomes from our method suggest that the EET, as in-

Table 5
Absolute percentage errors of Empirical Esscher Transform estimates for Petrobras

Maturity	Moneyness (spot/strike)		EET-BtG	EET-B	BSM	HN
T = 17/252	Deep-out-of-the-money	0.95	**30.0697**	51.6393	63.7784	77.4953
	Out-of-the-money	0.97	**39.0752**	56.8726	64.7611	78.4734
		0.98	**16.1753**	28.2949	32.8115	43.1060
	In-the-money	1.03	**1.6857**	7.5260	8.1260	13.4044
		1.07	**0.8618**	2.1482	2.0521	5.2937
	Deep-in-the-money	1.13	0.3616	0.3637	**0.1661**	1.8891
		1.17	**1.4873**	1.6451	1.5740	2.7965
		1.24	1.9716	1.9494	1.9618	**1.1296**
		1.31	**0.9664**	0.9686	0.9669	1.1198
		1.38	**0.7244**	0.7245	**0.7244**	0.7847
		1.61	**1.3303**	**1.3303**	**1.3303**	1.3654
T =40/252	Deep-out-of-the-money	0.88	97.0938	112.7758	140.7773	**91.3502**
		0.90	**81.8346**	90.5321	107.5664	91.6974
		0.94	**56.2463**	56.8326	64.2107	64.7549
		0.96	**29.6578**	32.5719	37.0929	39.6310
	Out-of-the-money	0.97	**17.9689**	23.0286	26.3649	29.5165
	At-the-money	1.02	**8.0226**	13.4987	14.6670	18.5379
	In-the-money	1.07	**2.9488**	6.5800	6.9186	10.3816
	Deep-in-the-money	1.13	**2.2976**	3.7873	3.7871	6.6295
		1.16	**1.4679**	2.2564	2.2069	4.6907
		1.24	**1.7016**	1.8856	1.8454	3.7674
		1.31	**3.8063**	3.8681	3.8496	5.4948
		1.37	**3.4178**	3.4406	3.4331	3.9858
T = 59/252	Deep-out-of-the-money	0.94	**40.0601**	47.5063	53.6051	54.6439
	At-the-money	1.02	19.8938	**19.0197**	20.5533	24.8240
	Deep-in-the-money	1.12	4.1748	**3.1370**	3.3037	6.7046
T = 121/252	Deep-out-of-the-money	0.96	**0.6364**	5.1689	3.9998	7.5215

This table contains the prices for a European call option from EET-BtG (empirical Esscher transform with Beta-t-GARCH model), EET-B (bootstrap with replacement on historical returns), BSM (Black-Scholes) and HN (Heston-Nandi) methods for different moneyness, maturities and they are compared to the true market price of Petrobras data. The numbers reported for each combination are the APEs. In the proposed method, we use 252 returns and the simulation is repeated 15,000 times.

Table 6
Absolute percentage errors of Empirical Esscher Transform estimates for Vale

Maturity	Moneyness (Spot/strike)		EET-BtG	EET-B	BSM	HN
T = 17/252	Deep-out-of-the-money	0.93	**132.3809**	177.7454	207.6516	149.4234
		0.95	**102.9132**	136.3859	154.5106	124.6248
	Out-of-the-money	0.98	**42.8648**	62.7852	69.9354	59.8857
	At-the-money	1.00	**15.7799**	29.4206	32.8052	28.5659
		1.00	**18.2929**	31.6931	34.8277	30.9905
		1.02	**0.6636**	9.1424	10.4689	9.1643
	In-the-money	1.05	**1.5949**	6.2268	6.6073	6.6472
		1.07	**7.6055**	11.3211	11.4716	11.8876
		1.11	**0.7302**	2.0628	1.9543	2.1187
		1.11	**0.0997**	1.0810	0.9691	1.8023
	Deep-In-the-money	1.14	1.4029	0.8775	0.9795	**0.1384**
		1.14	**0.8132**	1.2765	1.1766	2.0321
		1.18	**5.3185**	5.4862	5.4232	6.2524
		1.21	**0.7264**	0.7801	0.7516	1.4623
		1.36	1.8745	**1.8742**	1.8745	1.8421
		1.37	2.7025	**2.7022**	2.7025	2.6711
		1.47	**1.9758**	**1.9758**	1.9759	1.9513
T = 40/252	Deep-out-of-the-money	0.89	**97.1126**	198.1214	228.7324	140.4928
		0.89	**52.9349**	129.5083	152.6143	86.4836
		0.90	**98.3808**	182.5218	208.0594	138.9450
		0.93	**72.0110**	113.4649	126.1089	96.9817
		0.95	**21.8471**	42.4646	48.6054	35.7837
	Out-of-the-money	0.98	**17.2539**	30.8073	34.7675	72.3618
		0.98	**13.4745**	26.2884	30.0114	23.0940
	At-the-money	1.00	**8.7974**	18.5782	20.9070	17.1773
		1.01	**11.0570**	20.0682	22.0163	19.1798
	In-the-money	1.03	**2.7025**	9.9956	11.3534	9.7122
		1.08	**0.5307**	4.7083	5.1001	5.5257
		1.11	**2.7652**	5.3974	5.5660	6.4849
	Deep-In-the-money	1.12	**1.0646**	3.1720	3.2741	4.3086
		1.17	**2.4545**	3.4474	3.4372	4.7063
		1.29	**3.7298**	3.8515	3.8311	4.9972
T = 59/252	Deep-out-of-the-money	0.86	**22.6917**	119.5118	144.6253	64.5129
		0.93	**22.9524**	53.6051	61.1890	46.6326
		0.93	**23.9124**	53.7180	61.0375	47.2443
	At-the-money	1.00	**5.3325**	16.2658	18.5636	16.1516
	In-the-money	1.03	**3.1441**	9.9069	11.3773	10.5606

This table contains the prices for a European call option from EET-BtG (empirical Esscher transform with Beta-t-GARCH model), EET-B (bootstrap with replacement on historical returns), BSM (Black-Scholes) and HN (Heston-Nandi) methods for different moneyness, maturities and they are compared to the true market price of Vale data. The numbers reported for each combination are the APEs. In the proposed method, we use 252 returns and the simulation is repeated 15,000 times.

Table 7
Comparison between the absolute percentage errors of the proposed method for different sample sizes for Petrobras

Maturity	Moneyness (spot/strike)		EET-BtG	EET-B	EET-BtG*	EET-B*
T = 17/252	Deep-out-of-the-money	0.95	30.0697	51.6393	**28.8035**	52.4537
	Out-of-the-money	0.97	39.0752	56.8726	**35.4213**	57.5193
		0.98	16.1753	28.2949	**12.6480**	28.7008
	In-the-money	1.03	1.6857	7.5260	**1.2814**	7.6249
		1.07	0.8618	2.1482	**0.3616**	2.1653
	Deep-in-the-money	1.13	0.3616	0.3637	**0.1626**	0.3199
		1.17	**1.4873**	1.6451	1.5369	1.6473
		1.24	1.9716	1.9494	**1.9543**	1.9492
		1.31	**0.9664**	0.9686	0.9693	0.9687
		1.38	**0.7244**	0.7245	0.7247	0.7245
		1.61	**1.3303**	**1.3303**	**1.3303**	**1.3303**
T =40/252	Deep-out-of-the-money	0.88	**97.0938**	112.7758	118.6863	116.2486
		0.90	**81.8346**	90.5321	85.3186	92.6886
		0.94	56.2463	56.8326	**48.1147**	57.8006
		0.96	29.6578	32.5719	**25.2225**	33.1587
	Out-of-the-money	0.97	17.9689	23.0286	**16.4807**	23.4608
	At-the-money	1.02	**8.0226**	13.4987	9.2738	13.6603
	In-the-money	1.07	**2.9488**	6.5800	4.1976	6.6383
	Deep-in-the-money	1.13	**2.2976**	3.7873	2.7985	3.8054
		1.16	**1.4679**	2.2564	1.7425	2.2653
		1.24	**1.7016**	1.8856	1.7912	1.8884
		1.31	**3.8063**	3.8681	3.8544	3.8692
		1.37	**3.4178**	3.4406	3.4420	3.4408
T = 59/252	Deep-out-of-the-money	0.94	**40.0601**	47.5063	43.4797	48.4873
	At-the-money	1.02	19.8938	19.0197	**15.9649**	19.2947
	Deep-in-the-money	1.12	4.1748	3.1370	**2.1643**	3.1809
T = 121/252	Deep-out-of-the-money	0.96	**0.6364**	5.1689	1.2779	2.1500

This table contains the prices for a European call option from EET-BtG (empirical Esscher transform with Beta-t-GARCH model) and the EET-B (bootstrap with replacement on historical returns) method for different moneyness, maturities and they are compared to the true market price of Petrobras data. The numbers reported for each combination are the APEs. We use 252 returns and the simulation is repeated 15,000 times in prices EET-BtG. We use 50,000 returns and the simulation is repeated 200 times in prices EET-BtG*.

Table 8
Comparison between the absolute percentage errors of the proposed method for different sample sizes for Vale

Maturity	Moneyness (Spot/strike)		EET-BtG	EET-B	EET-BtG*	EET-B*
T = 17/252	Deep-out-of-the-money	0.93	**132.3809**	177.7454	129.1951	180.9486
		0.95	102.9132	136.3859	**92.3902**	138.1864
	Out-of-the-money	0.98	42.8648	62.7852	**36.1791**	63.3947
	At-the-money	1.00	15.7799	29.4206	**12.6889**	29.6875
		1.00	18.2929	31.6931	**15.4971**	31.9417
		1.02	0.6636	9.1424	**0.1448**	9.2528
	In-the-money	1.05	1.5949	6.2268	**1.3282**	6.2675
		1.07	**7.6055**	11.3211	7.6805	11.3457
		1.11	**0.7302**	2.0628	1.0069	2.0680
		1.11	**0.0997**	1.0810	0.1570	1.0852
	Deep-In-the-money	1.14	1.4029	0.8775	1.2766	**0.8763**
		1.14	**0.8132**	1.2765	0.9220	1.2770
		1.18	**5.3185**	5.4862	5.3597	5.4856
		1.21	**0.7264**	0.7801	0.7432	0.7797
		1.36	1.8745	1.8742	**1.8740**	1.8742
		1.37	2.7025	2.7022	**2.7021**	2.7022
		1.47	**1.9758**	**1.9758**	1.9759	**1.9758**
T = 40/252	Deep-out-of-the-money	0.89	**97.1126**	198.1214	214.8283	203.1444
		0.89	**52.9349**	129.5083	141.1287	133.2440
		0.90	**98.3808**	182.5218	188.8369	186.5764
		0.93	**72.0110**	113.4649	104.8281	115.2330
		0.95	**21.8471**	42.4646	34.2552	43.2622
	Out-of-the-money	0.98	**17.2539**	30.8073	22.8009	31.2986
		0.98	**13.4745**	26.2884	18.5726	26.7483
	At-the-money	1.00	**8.7974**	18.5782	11.9412	18.8487
		1.01	**11.0570**	20.0682	13.7751	20.2916
	In-the-money	1.03	**2.7025**	9.9956	4.8394	10.1493
		1.08	**0.5307**	4.7083	2.1299	4.7574
		1.11	**2.7652**	5.3974	3.7281	5.4220
	Deep-In-the-money	1.12	**1.0646**	3.1720	1.8532	3.1884
		1.17	**2.4545**	3.4474	2.8787	3.4521
		1.29	**3.7298**	3.8515	3.7985	3.8518
T = 59/252	Deep-out-of-the-money	0.86	**22.6917**	119.5118	148.5000	125.0559
		0.93	**22.9524**	53.6051	50.7790	55.1348
		0.93	**23.9124**	53.7180	50.5609	55.1857
	At-the-money	1.00	**5.3325**	16.2658	12.1033	16.6808
	In-the-money	1.03	**3.1441**	9.9069	6.3625	10.1655

This table contains the prices for a European call option from EET-BtG (empirical Esscher transform with Beta-t-GARCH model) and the EET-B (bootstrap with replacement on historical returns) method for different moneyness, maturities and they are compared to the true market price of Vale data. The numbers reported for each combination are the APEs. We use 252 returns and the simulation is repeated 15,000 times in prices EET-BtG. We use 50,000 returns and the simulation is repeated 200 times in prices EET-BtG*.

troduced by Pereira and Veiga (2017), warrants further exploration. It would be beneficial to extend this research by encompassing a broader range of assets and datasets. Additionally, testing this method within real-world trading scenarios would be of significant value.

References

Alcock, J. and Carmichael, T. (2008). Nonparametric American option pricing, *Journal of Futures Markets* **28**(8): 717–748.

Almeida, C. and Azevedo, R. (2022). Nonparametric option pricing with generalized entropic estimators, *SSRN Electronic Journal* .

Badescu, A., Elliott, R., Kulperger, R., Miettinen, J. and Siu, T. K. (2011). A comparison of pricing kernels for GARCH option pricing with generalized hyperbolic distributions, *International Journal of Theoretical and Applied Finance* **14**(5): 669–708.

Badescu, A. and Kulperger, R. (2007). GARCH option pricing: A semiparametric approach, *Insurance: Mathematics and Economics* **43**(1): 69–84.

Bakshi, G., Cao, C. and Chen, Z. (1997). Empirical performance of alternative option pricing models, *Journal of Finance* **52**: 2003–2049.

Barone-Adesi, G., Engle, R.-F. and Mancini, L. (2008). A GARCH option pricing model with filtered historical simulation, *Review of Financial Studies* **21**: 1223–1256.

Bates, D. (2000). Post-’87 crash fears in S&P 500 futures options, *Journal of Econometrics* **94**(1–2): 181–238.

Binghan, N. and Kiesel, R. (2004). *Risk-Neutral Valuation: Pricing and Hedging of Financial Derivatives*, Springer.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities, *The Journal of Political Economy* **81**: 637–654.

Bollerslev, T. (1987). A conditionally heteroskedastic time series model for speculative prices and rates of return, *Review of Economics and Statistics* **69**: 542–547.

Bollerslev, T. and Mikkelsen, H. (1996). Modeling and pricing long memory in stock market volatility, *Journal of Econometrics* **73**: 151–184.

Broadie, M., Chernov, M. and Johannes, M. (2007). Model specification and risk premia: Evidence from futures options, *Journal of Finance* **62**: 1453–1490.

Chernov, M., Gallant, A. R., Ghysels, E. and Tauchen, G. (2003). Alternative models for stock price dynamics, *Journal of Econometrics* **116**: 225–257.

Chorro, C., Guegan, D. and Ielpo, F. (2008). Option pricing under GARCH models with generalized hyperbolic innovations (II): Data and results, *Documents de Travail du Centre d'Economie de la Sorbonne* .

Christoffersen, P., Jacobs, K. and Chang, K. (2006). Volatility components: affine restrictions and nonnormal innovations. Working paper, McGill University.

Christoffersen, P., Jacobs, K. and Heston, S. (2011). A GARCH option model with variance-dependent pricing kernel. Working Paper. University of Toronto. Canada.

Cox, J. C. and Ross, S. A. (1976). The valuation of options for alternative stochastic processes, *Journal of Financial Economics* **3**: 145–146.

Creal, D., Koopman, S. and Lucas, A. (2013). Generalized autoregressive score models with applications, *Journal of Applied Econometrics* **28**: 777–95.

Duan, J. (1995). The GARCH option pricing model, *Mathematical Finance* **5**(1): 13–32.

Duan, J. (2002). Nonparametric option pricing by transformation. Working Paper, Rotman School of Management.

Engle, R. and Lee, G. (1999). A long-run and short-run component model of stock return volatility in cointegration, causality and forecasting, *in* R. Engle and H. White (eds), *Cointegration, Causality and Forecasting*, Oxford University Press.

Eraker, B., Johannes, M. and Polson, N. (2003). The impact of jumps in volatility and returns, *The Journal of Finance* **58**: 1269–1300.

Esscher, F. (1932). On the probability function in the collective theory of risk, *Skandinavisk Aktuarietidskrift* **15**: 175–195.

Galai, D. (1979). A convexity test for traded options, *Quaterly Review of Economics and Business* **19**: 83–90.

Gencay, R. and Salih, A. (2003). Degree of mispricing with the Black-Scholes model and nonparametric cures, *Annals of Economic and Finances* **4**: 73–101.

George, T. J. and Longstaff, F. A. (1993). Bid-ask spreads and trading activity in the S&P 100 index options market, *Journal of Financial and Quantitative Analysis* **28**: 381–398.

Gerber, H. and Shiu, E. (1994). Option pricing by Esscher transforms, *Transactions of Society of Actuaries* **46**: 99–191.

Gray, P. and Newman, S. (2005). Canonical valuation of options in the presence of stochastic volatility, *Journal of Futures Markets* **25**(1): 1–19.

Hagan, P. and West, G. (2006). Interpolation methods for yield curve construction, *Applied Mathematical Finance* **13**(2): 89–129.

Haley, R. and Walker, T. (2010). Alternative tilts for nonparametric option pricing, *Journal of Future Markets* **30**(10): 983–1006.

Harrison, J. M. and Kreps, D. M. (1979). Martingale and arbitrage in multiperiod securities markets, *Journal of Economic Theory* **20**: 381–408.

Harrison, J. M. and Pliska, S. R. (1981). Martingales and stochastic integrals in the theory of continuous trading, *Stochastic Processes and Theirs Applications* **11**: 215–280.

Harvey, A. C. (2013). *Dynamic Models for Volatility and Heavy Tails*, Econometric Society Monograph, Cambridge University Press.

Heston, S. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options, *Review of Financial Studies* **6**(2): 327–343.

Heston, S. L. and Nandi, S. (2000). A closed-form GARCH option valuation model, *Review of financial Studies* **13**(3): 585–625.

Jackwerth, J. C. (2004). Option-implied risk-neutral distribution and risk aversion, *The Research Foundation of AIMR (Association for Investment Management and Research)* .

Liu, Y., Li, J. S.-H. and Ng, A. C.-Y. (2015). Option pricing under GARCH models with hansen's skewed-t distributed innovations, *North American Journal of Economics and Finance* **31**: 108–125.

Mann, H. B. and Wald, A. (1943). On stochastic limit and order relationships, *Annals of Mathematical Statistics* **14**(3): 217–226.

Menn, C. and Rachev, S. (2005). Smoothly truncated stable distributions, GARCH-models, and option pricing. Working paper, Cornell University and University of Karlsruhe and UCSB.

Meyn, S. P. and Tweedie, R. L. (1994). State-dependent criteria for convergence Markov chains, *Ann. Appl. Probab.* **4**: 149–168.

Nelson, D. (1996). Asymptotically optimal smoothing with ARCH models, *Econometrica* **64**: 561–573.

Nelson, D. B. and Cao, C. (1992). Inequality constraints in the univariate GARCH model, *Journal of Business and Economic Statistics* **10**: 229–235.

Pereira, M. and Veiga, A. (2017). Nonparametric estimation of risk-neutral distribution via the empirical Esscher transform, *Brazilian Finance Review* **15**(2): 167–195.

Pérignon, C. and Villa, C. (2002). Extracting information from option markets: Smiles, state-price densities and risk aversion, *European Financial Management* **8**: 495–513.

Satoyoshi, K. and Mitsui, H. (2011). Empirical study of Nikkei 225 options with the Markov switching GARCH model, *Asia-Pacific Financial Markets* **18**: 55–68.

Siu, T. K., Tong, H. and Yang, H. (2004). On pricing derivatives under GARCH models: A dynamic Gerber-Shiu approach, *North American Actuarial Journal* **8**(3): 17–31.

Stutzer, M. (1996). A simple nonparametric approach to derivative security valuation, *Journal of Finance* **51**(5): 1633–52.

Tsay, R., S. (2013). *An introduction to Analysis of Financial Data with R*, 1st ed., Wiley.