Using hierarchical risk parity in the Brazilian market: An out-of-sample analysis

Felipe Reis[†]
Anderson Sobreira[‡]
Carlos Trucíos[*]
Boris Asrilhant[**]

Abstract Portfolio allocation is an important tool for portfolio managers and investors interested in diversification as well as improvements in out-of-sample portfolio performance. Recently, new portfolio allocation strategies based on unsupervised machine learning have been proposed in the literature, with hierarchical risk parity being one of the most popular. This article uses assets from the Brazilian financial market to perform an extensive out-of-sample comparison of hierarchical risk parity against widely-known, traditional portfolio allocation techniques. The results suggest that, in general, hierarchical risk parity does not report the best performance but, in some performance measures, performs equally well to other approaches. Overall, hierarchical risk parity outperforms the market index.

Keywords: Asset allocation; Diversification; Machine Learning; Markowitz; Portfolio selection.
JEL Code: C52, C58, G11.

1. Introduction

Portfolio allocation plays an important role in finance for both portfolio managers and individual investors, offering a means to diversify investments and potentially improve the out-of-sample portfolio performance. Over the years, various approaches have been developed and proposed in the literature to address the challenge of allocating portfolios. These approaches range from simple methodologies, such as equally-weighted portfolios, to more complex ones, such as those proposed by Markowitz (1952), Qian (2005), Choueifaty and Coignard (2008), Maillard et al. (2010), Lopez de Prado (2016), among others.

Submitted on August 5, 2023. Accepted on August 30, 2023. Published online in December 2023. Editor in charge: Bruno Giovannetti.

[†]Universidade Federal do Rio de Janeiro, Brazil: felipereis150@gmail.com
[‡]Universidade Federal do Rio de Janeiro, Brazil: andersong.sobreira@gmail.com
[*]Universidade Estadual de Campinas, Brazil: ctrucios@unicamp.br
[**]Universidade Federal do Rio de Janeiro, Brazil: boris@facc.ufrj.br

The easiest, fastest portfolio allocation strategy is the equally-weighted technique, which does not require solving an optimization problem and allocates the same amount of wealth in each asset. This approach follows the principle of not putting all eggs in one basket, and can be appropriate when neither the risks nor the expected returns can be forecasted or when the estimation error is large (De Carvalho et al., 2012; Pflug et al., 2012; Battaglia and Leal, 2017).

Another approach, the traditional, most widely-known portfolio allocation technique, was put forth by Markowitz (1952), which is nowadays called as modern portfolio theory, mean-variance portfolio or even Markowitz portfolio optimization. Markowitz portfolio optimization aims to build portfolios that maximize the expected return for a given level of risk or, in case of its dual problem, minimize risk for a given level of expected return (Meucci, 2007). This approach is broadly used for both academic and practitioners, although, despite its widespread use, Markowitz portfolio optimization has faced criticisms and limitations; see, for instance, Michaud (1989), Becker et al. (2015), Huang and Yu (2020), and Oliveira et al. (2023) to quote only a few.

The main criticism of Markowitz portfolio optimization relies on the fact that this approach needs to meet requirements that never hold in practice: the true mean vector and the covariance matrix of asset returns should be known. In empirical applications it is necessary to estimate them from data, implying in some degree of estimation error which, as pointed out by Michaud (1989), Chopra et al. (1993) and Chopra and Ziemba (1993), may affect the portfolio weights obtained by the Markowitz portfolio optimization. This estimation error yields to extreme and/or unrealistic portfolio weights, lack of diversification and poor out-of-sample performance (Michaud, 1989; Wolf, 2004; Becker et al., 2015; Huang and Yu, 2020), which are undesirable features in portfolio allocation.

Alternatively, an approach that has drawn attention for both academic and practitioners is the risk parity portfolio (Qian, 2005) in which, regardless the portfolio risk measure, the marginal risk contribution to every asset in the portfolio is set to be equal. One of the main characteristics of this strategy refers to well-diversified portfolio weights. For instance, comparatively to the Markowitz portfolio weights, which tend to be concentrated in a few assets, risk parity portfolio weights tend to be spread out among all of them, yielding to more diversified portfolios and limiting the impact of large losses from individual assets (Chaves et al., 2011; Bai et al., 2016).

Since the proposals of Markowitz (1959) and Qian (2005) popularized quantitative portfolio allocation techniques, several other portfolio allocation strategies have been proposed in the literature, which also attracted the attention of both academics and practitioners (De Carvalho et al., 2012; Ardia, Bolliger, Boudt and Gagnon-Fleury, 2017; Nakagawa et al., 2018; du Plessis and van Rensburg, 2020), such as the inverse-volatility weighted portfolio of De Carvalho et al. (2012), the maximum decorrelation of Christoffersen et al. (2012) and the maximum diversification of Choueifaty and Coignard (2008).

Recently, Lopez de Prado (2016) proposed the hierarchical risk parity (HRP) portfolio, a portfolio allocation strategy that avoids inverting the covariance matrix (minimizing the effect of estimation error) and relies on an unsupervised machine learning problem (clusterization) rather than optimization, to obtain the portfolio weights. This approach explores an earlier idea of Simon (1962), who argued that complex systems, such as financial markets, are usually organized in a hierarchical manner.

HRP portfolios bring a new, innovative way to deal with portfolio allocation. This procedure has been applied in different markets with encouraging results (Burggraf, 2021; Nourahmadi and Sadeqi, 2022; Sen et al., 2021; Sen and Dutta, 2022) but its effectiveness in the Brazilian market has not been well established yet. In order to investigate its effectiveness, this study performs an out-of-sample comparison of HRP portfolio against other well-known portfolio allocation techniques using data from the Brazilian stock market. Therefore, this study aims to assess whether HRP offers superior out-of-sample portfolio performance compared to other approaches.

This comparison contributes to the literature to a better understanding in which situations HRP portfolios could be an interesting approach to be implemented, as well as to evaluate its performance using real data. Additionally, the results obtained provide portfolio managers and individual investors with a comprehensive out-of-sample analysis to make a decision whether HRP is worth to be applied in the Brazilian market.

The remaining part of the paper is organized as follows. Section 2 presents a brief review of empirical applications of portfolio allocation strategies in the Brazilian market and also places this work in context. Section 3 describes the portfolio allocation strategies implemented in this paper, with special emphasis on HRP. Section 4 describes the data set used and presents the out-of-sample comparison. Finally, Section 5 points out the final considerations.

2. Literature Review

There are several papers in the literature addressing portfolio allocation strategies in the Brazilian market. However, most of them are based on classical portfolio allocation procedures and do not consider hierarchical risk parity, which is the focus of this paper.

Among the papers addressing portfolio allocation, Farias et al. (2006) compares the minimum variance portfolio against the mean absolute deviation (Konno and Yamazaki, 1991) and minimax (Young, 1998) strategies in a daily data context and concludes that the minimax strategy usually performs better than its competitors. Santos and Tessari (2012) using daily data and considering mean-variance, minimum variance, equally-weighted portfolios and the market index, compares the out-of-sample performance of those strategies using different covariance matrix estimators, and concludes that, in general, strategies based on optimization perform better than the equally-weighted portfolios and the market index. Rubesam et al. (2013) and Caldeira et al. (2013), both using daily data, compare several covariance matrix estimators into a minimum variance portfolio allocation problem, where the former concludes that simplest covariance matrix estimators yields to better out-of-sample performance and the latter concludes that the estimator of Santos and Moura (2014) leads to better out-of-sample performance. Naibert and Caldeira (2015), using daily data, compares the mean-variance and minimum variance portfolios with and without portfolio weights restrictions as proposed by Fan et al. (2012), and the results evidence that restrictions can be very useful to improve portfolio performance. Borges et al. (2015) and Caldeira et al. (2017) compare the minimum variance portfolio using several intraday covariance matrix estimators, concluding that covariance matrices estimated using intraday data leads to substantial improvements in comparison with covariance matrices based on daily data. Bortoluzzo et al. (2018), using daily data, compares risk parity against the minimum variance and equally weighted portfolios, the results indicating that the minimum variance portfolio achieved the best performance in terms of risk. Oliveira et al. (2023), using monthly data, compares Markowitz optimizations against their re-sampling portfolio (Michaud and Michaud, 1998) alternatives and concludes that re-sampling portfolio yields to better out-of-sample performance. Numerous other works such as Iquiapaza et al. (2016), Leal and Campani (2016), Souza et al. (2017), Maciel (2021), Pereira and Oliveira (2021) also deal with portfolio selection in the Brazilian scenario but none of them used HRP portfolios.

The only works dealing with the use of HRP in the Brazilian market are Duarte and De Castro (2020) and Freitas and Junior (2023). The former uses daily data and compares HRP against minimum variance portfolio and the market index, and the results indicate that although HRP outperforms the minimum variance portfolio in terms of average portfolio returns, it is outperformed in terms of risk. The latter, also using daily data, concludes that HRP only outperforms the Brazilian market index (Ibovespa) in terms of Sharpe ratio.

Therefore, the present work can be seen as complementary to the aforementioned ones. First, this comparison is based on monthly rather than on daily data. Monthly data implies in smaller transactions costs due to rebalancing frequency and, since monthly returns has no (or very weak) ARCH effects, simple covariance matrix estimators can be used.[1] Second, beyond the equally-weighted and minimum variance portfolios, this comparison includes a broad number of well-known portfolio allocation techniques, most of them were not considered in the previous comparisons. Third, this comparison considers other portfolio performance measures rather than the classical Sharpe ratio and annualized volatility. Additionally, when applicable, hypothesis testing is used to verify statistically significant differences, a procedure which is not performed in the previous works.

3. Portfolio allocation strategies

Let $P_{i,t}$ be the closing price of asset i at time t and let $r_{i,t} = 100 \times (P_{i,t} - P_{i,t-1})/P_{i,t}$ be its corresponding simple return in percentage. Let $\mathbf{r}_t = (r_{1,t}, \ldots, r_{N,t})$ and $\boldsymbol{\omega} = (\omega_1, \cdots, \omega_N)$ be a N-dimensional vector of returns and its corresponding N-dimensional vector of portfolio weights, respectively. Then, the portfolio return at time t is given by

$$R_t = \mathbf{r}_t \boldsymbol{\omega}_t' = \omega_1 r_{1,t} + \cdots + \omega_N r_{N,t}.$$

In this section, six well-known methods to obtain $\boldsymbol{\omega} = (\omega_1, \cdots, \omega_N)$ will be briefly introduced, namely, the equally-weighted, minimum variance, risk parity, maximum diversification, maximum decorrelation and inverse-volatility portfolios. Additionally, the recently proposed HRP portfolio of Lopez de Prado (2016) is also described.

3.1 Equally-weighted (EW) portfolio

Also called naive or $1/N$ strategy, the equally-weighted portfolio is the simplest method to allocate wealth across assets. It is not grounded on complex calculations and no optimization problem is considered. For a set of N

[1] The other two paper mentioned ignore the daily evolution of variances and covariances and use simple covariance matrix estimators. For a better comparison, simple covariance matrix estimators should be used with monthly data or multivariate volatility models should be used with daily data. In this paper, we address the former and left the latter as a future comparison.

assets, the portfolio weights are defined by

$$\omega_i = 1/N, \quad \text{for } i = 1, \cdots, N.$$

Despite its simplicity, this method is advocated by Duchin and Levy (2009), DeMiguel et al. (2009), Malladi and Fabozzi (2017), among others, to outperform complex alternatives. However, recent studies reveal that this finding is not always hold (see; for instance, Fugazza et al., 2015; Engle et al., 2019; De Nard et al., 2021; Trucíos et al., 2023) and is usually outperformed by other approaches. This method is the primary benchmark in empirical applications.

3.2 Minimum variance (MV) portfolio

Markowitz optimization (Markowitz, 1952, 1959) yields to portfolios in the efficient frontier. This approach, also known as mean-variance portfolio, requires such input parameters as the mean vector and the covariance matrix. As the mean vector is difficult to be estimated with some degree of accuracy (Merton, 1980), a portfolio that only requires the covariance matrix as an input parameter, called minimum variance portfolio, is then preferable. In this portfolio allocation strategy, the weights are obtained by minimizing

$$\boldsymbol{\omega}\Sigma\boldsymbol{\omega}', \quad \text{subject to} \sum_{i=1}^{N} \omega_i = 1,$$

where Σ is the covariance matrix of asset returns. Often, due to investor preferences and/or financial institution restrictions, no short-selling constraints ($\omega_i \geqslant 0$ for $i = 1, \cdots, N$) are included in the optimization problem.

3.3 Risk parity portfolio (RP)

Advocated by Qian (2005, 2006, 2011, 2016), risk parity portfolios became very popular and quickly adopted by portfolio managers due to its superior performance in empirical data (Qian, 2013). This approach seeks portfolios where marginal risk contribution of every asset in the portfolios are equal. Consequently, assets with lower risk will have larger allocation than assets with higher risk. Using the portfolio variance as a measure of risk, the portfolio weights are obtained by minimizing

$$\sum_{i=1}^{N} (\%RC_i - 1/N)^2,$$

where $\%RC_i = \omega_i(\Sigma\boldsymbol{\omega}')_i/\boldsymbol{\omega}\Sigma\boldsymbol{\omega}'$ is the percentage of variance contribution of asset i to the entire portfolio and $(\Sigma\boldsymbol{\omega}')_i$ stands for the ith position of the vector $\Sigma\boldsymbol{\omega}'$.

3.4 Maximum diversification (MD) portfolio

The maximum diversification (MD) portfolio allocation strategy introduced by Choueifaty and Coignard (2008) and developed by Choueifaty et al. (2013), also known as the most diversified portfolio, seeks portfolio weights that maximize the diversification ratio D, which is calculated by the weighted average of each volatility divided by the total portfolio volatility. Thus, the portfolio weights are obtained by maximizing

$$D = \frac{\boldsymbol{\omega} \times (\sigma_1, \cdots, \sigma_N)'}{\sqrt{\boldsymbol{\omega}\Sigma\boldsymbol{\omega}'}}, \quad \text{subject to} \sum_{i=1}^{N} \omega_1 = 1 \text{ and } \omega_i \geqslant 0.$$

Note that, in the extreme case of a portfolio of a single asset (mono-asset portfolio), the diversification ratio is equal to one and, as a consequence, the portfolio is poorly diversified.

3.5 Maximum decorrelation (MDE) portfolio

Introduced by Christoffersen et al. (2012) and Goltz and Sivasubramanian (2018), this portfolio allocation strategy is a particular case of the minimum variance portfolio in which the covariance matrix Σ is replaced by the correlation matrix ρ. Thus, portfolio weights are obtained by minimizing

$$\boldsymbol{\omega}\rho\boldsymbol{\omega}', \quad \text{subject to} \sum_{i=1}^{N} \omega_i = 1.$$

Usually similar to other portfolio allocation strategies, no short-selling constraints are also imposed in the optimization problem. This strategy explores low correlations among assets.

3.6 Inverse-volatility weighted portfolio (IV)

Proposed by De Carvalho et al. (2012), the inverse-volatility weighted portfolio does not take covariances into account and allocates assets accordingly to its proportion of inverse volatility over the volatility harmonic mean of all assets considered, that is, the portfolio weights are obtained through

$$\boldsymbol{\omega} = \left(\sum_{j=1}^{N} 1/\sigma_j \right)^{-1} \times (1/\sigma_1, \cdots, 1/\sigma_N).$$

3.7 Hierarchical risk parity (HRP) portfolio

Finally, HRP is a new, innovative portfolio allocation technique proposed by Lopez de Prado (2016). This approach uses the information provided by the covariance matrix without inverting it, being then an attractive alternative in case the covariance matrix is numerically ill-conditioned. The procedure can be summarized in the following steps:

1. Let $\rho = \{\rho_{ij}\}_{i,j=1,\cdots N}$ be the $N \times N$ correlation matrix with elements ρ_{ij}. Compute the matrix $D = \{d_{ij}\}_{i,j=1,\cdots N}$, where $d_{ij} = \sqrt{0.5 \times (1-\rho_{ij})}$, is a measure of distance between assets i and j.

2. Calculate the $N \times N$ matrix $\tilde{D} = \{\tilde{d}_{ij}\}_{i,j=1,\cdots,N}$ with elements $\tilde{d}_{ij} = \sqrt{\sum_{k=1}^{N}(d_{ki} - d_{kj})^2}$ (the Euclidean distance between columns i and j of matrix D) and apply the single linkage hierarchical clustering procedure using $\tilde{D}$ as distance matrix.

3. Ordering the assets according to the dendrogram obtained by the clusterization in the previous step.

4. Denote by $L = \{L_1\} = \{C_1, \cdots, C_N\}$ the cluster with all (ordering) assets and by $|L_1|$ its number of elements. Assign unity weights to all elements in L_1, i.e, $\omega_i = 1$ for $i = 1, \cdots, N$.

5. For each $L_i \in L$ such that $|L_i| > 1$:

   (a) Preserving the ordering, split L_i into $L_i^{(1)}$ and $L_i^{(2)}$, such that $L_i^{(1)} \cup L_i^{(2)} = L_i$ and $|L_i^{(1)}| = \lfloor 0.5|L_i| \rfloor$.

   (b) For $k = 1, 2$, define $V_i^{(k)} = \boldsymbol{\omega}^{(k)} \Sigma_i^{(k)} \boldsymbol{\omega}^{(k)\prime}$, where $\Sigma_i^{(k)}$ is the covariance matrix of the elements of $L_i^{(k)}$ and

   $$\boldsymbol{\omega}^{(k)} = \frac{1}{Trace(Diag(\Sigma_i^{(k)})^{-1})} \times Diag(\Sigma_i^{(k)})^{-1}.$$

   (c) Compute $\alpha = 1 - \frac{V_i^{(1)}}{V_i^{(1)} + V_i^{(2)}}$ and update the portfolio weights by $\boldsymbol{\omega}^{(1)} = \alpha \boldsymbol{\omega}^{(1)}$ and $\boldsymbol{\omega}^{(2)} = (1-\alpha)\boldsymbol{\omega}^{(2)}$, respectively.

   (d) $L = \{L_i^{(1)}, L_i^{(2)}\}$

4. Empirical application

The portfolio allocation strategies described in Section 3 were applied to Brazilian financial assets and their out-of-sample portfolio performances were compared. Analysis considered two different window sizes in a rolling window scheme ($T = 60$ and $T = 120$, which correspond from five to ten years of monthly data, respectively) as well as two out-of-sample periods (full and before the COVID-19 pandemic) were considered. All analyses were performed in R software (Team, 2022) using the packages RiskPortfolios and HierPortfolios of Ardia, Boudt and Gagnon-Fleury (2017) and Trucios (2021), respectively. For reproduction purposes, the codes are freely available in the GitHub repository `https://github.com/felipereis150/BrazilianHRP`

4.1 Data

This study considered monthly returns spanning from January 2000 to June 2022 of assets traded in the Brazilian stock market (B3) and listed in the IBrX-100 index in July 2022. Series with missing values were excluded from the analysis, ending up with 25 assets over 270 months. All stock price data were downloaded from Economatica.

Descriptive statistics for these assets returns are reported in Table 1 and show that all returns have positive mean and kurtosis larger than 3, minimum values between -27% and -55% and maximum values between 28% and 104%, approximately. The largest monthly return is reported by USIM5 (Usinas Siderúrgicas de Minas Gerais - Usiminas) and the smallest one by LIGT3 (Light). USIM5 is also the most volatile asset with an annualized standard deviation of 59.3% ($\sqrt{12} \times 17.114$) while VIVT3 (Telefônica Brasil) is the less volatile asset with an annualized standard deviation of 26.5%, all but ABEV3 (Ambev) and SBSP3 (Companhia de Saneameno Básico do Estado de São Paulo - SABESP) have positive skewness. The most correlated assets (0.972) are PETR3 and PETR4 (Petrobras) and the less correlated ones (0.037) are EGIE3 (Engie Brasil) and VALE3 (Vale).

4.2 Out-of-sample evaluation

The out-of-sample portfolio performance was evaluated through economic measures as seen in Santos and Tessari (2012), Lopez de Prado (2016), Gambacciani and Paolella (2017), Trucíos et al. (2019), Oliveira et al. (2023), among others. Specifically, following the financial econometric literature, six well-known economic measures are used, briefly described as follows:

Table 1
Descriptive statistics of monthly returns of 25 stocks in the Brazilian stock market over the full sample period

	Min.	Max.	Mean	Std.	Skew.	Kurt.
ABEV3	−39.29	30.85	1.77	7.83	−0.27	6.28
ALPA4	−38.94	44.65	2.50	11.06	0.05	4.16
BBAS3	−40.18	47.54	2.14	11.98	0.17	4.56
BBDC3	−31.03	35.73	1.69	9.67	0.40	3.62
BBDC4	−31.93	30.43	1.69	9.83	0.19	3.42
BRKM5	−35.96	57.67	1.79	13.42	0.57	4.05
CMIG4	−36.18	54.05	1.63	10.63	0.37	5.54
CPLE6	−33.74	28.68	1.41	9.96	0.01	3.24
CSNA3	−50.23	83.78	2.56	15.85	0.52	5.28
EGIE3	−29.66	56.18	2.19	9.40	1.47	9.45
ELET3	−32.96	60.50	1.68	14.48	0.85	4.88
ELET6	−40.25	44.48	1.69	12.99	0.42	4.20
EMBR3	−43.77	39.14	1.01	11.64	0.07	5.04
GGBR4	−40.53	84.70	2.03	13.18	0.84	8.38
GOAU4	−45.06	100.83	1.91	14.08	1.07	11.91
ITSA4	−26.86	31.47	1.72	8.68	0.03	3.50
ITUB4	−27.81	36.10	1.61	9.12	0.08	3.92
LIGT3	−55.51	52.56	0.31	13.53	0.44	5.26
PETR3	−47.92	48.75	1.76	12.00	0.23	4.81
PETR4	−44.79	62.45	1.72	11.93	0.39	5.84
SBSP3	−34.93	37.68	1.52	10.50	−0.09	4.21
TIMS3	−28.84	65.51	1.31	12.07	0.93	6.28
USIM5	−45.16	104.54	2.17	17.11	1.19	9.22
VALE3	−27.78	31.40	2.06	10.25	0.30	3.43
VIVT3	−28.62	41.26	1.42	7.65	0.70	6.95

- **Annualized standard deviation (SD)**: is given by $\sqrt{12} \times \hat{\sigma}_p$, where $\hat{\sigma}_p$ is the sample standard deviation of the realized out-of-sample portfolio returns. The smaller the SD, the less risky the portfolio and, consequently, the better the portfolio performance.

- **Annualized Sharpe ratio (SR)**: is given by $\sqrt{12} \times SR$, where SR is the Sharpe ratio (Sharpe, 1975), a risk-adjusted performance measure defined by

$$SR = \frac{\bar{R}_p - \bar{R}_f}{\hat{\sigma}_{p-f}},$$

where $\bar{R}_f$ is the average risk-free rate, $\bar{R}_p$ is the sample mean of the out-of-sample realized portfolio returns and $\hat{\sigma}_{p-f}$ is the estimated standard deviation of the realized out-of-sample excess returns. The higher the annualized SR, the better the portfolio performance.

- **Annualized adjusted Sharpe ratio (ASR)**: is given by $\sqrt{12} \times ASR$, where ASR is the adjusted Sharpe ratio (Pézier and White, 2008), a risk-adjusted performance measure that penalizes the Sharpe ratio by negative skewness and an excess of kurtosis. It is given by

$$ASR = SR\left[1 + \left(\frac{\mu_3}{6}\right)SR - \left(\frac{\mu_4 - 3}{24}\right)SR^2\right],$$

where μ_3 and μ_4 stand for the skewness and kurtosis of the out-of-sample portfolio returns. The higher the ASR, the better the portfolio performance.

- **Annualized Sortino ratio (SO)**: is given by $\sqrt{12} \times SO$, where SO is the Sortino ratio (Sortino and Van Der Meer, 1991), another risk-adjusted performance measure. It is defined by

$$SO = \frac{\bar{R}_p}{\sqrt{\frac{\sum_{i=1}^{K} \min(0, R_{p,i} - MAR)^2}{K}}},$$

where K is the size of the out-of-sample period and MAR is the minimum accepted return, which is equal to the monthly risk-free rate. The higher the SO, the better the portfolio performance.

- **Average portfolio turnover (TO)**: measures the impact of transaction costs on portfolio performance per rebalancing, on average. It is given by

$$TO = \frac{1}{K-1} \sum_{i=2}^{K} \sum_{j=1}^{N} |\hat{\omega}_{i,j} - \hat{\omega}_{i,j}^{+}|,$$

where $\hat{\omega}_{i-1}^{+}$ stands for the one-step-ahead portfolio weights obtained in the window $i-1$ updated at time i before rebalancing to $\hat{\omega}_i$. Lower values of TO indicate smaller impacts of transaction costs on portfolio performance.

- **Sum of squared portfolio weights (SSPW)**: measures the diversification of the portfolio. The SSPW (Goetzmann and Kumar, 2008) is given by

$$SSPW = \frac{1}{K} \sum_{i=1}^{K} \sum_{j=1}^{N} \hat{\omega}_{i,j}^{2},$$

where $\hat{\omega}_{i,j}$ stands for the one-step-ahead portfolio weights obtained in the window i for asset j. Lower values of SSPW indicate higher levels of diversification.

In addition to the six measures described above, the bootstrap tests of Ledoit and Wolf (2008) and Ledoit and Wolf (2011) are used to verify whether the SR and the SD of HRP are statistically superior to the other methods, respectively. This article considered the value of the monthly risk-free rate as 0.005, which is compatible with the monthly Brazilian interest rate (CDI, in Portuguese).[2]

4.3 Results

Results for the out-of-sample analysis are reported in Table 2. The top and bottom panels report the results considering window sizes of $T = 60$ and $T = 120$ months, respectively. The left and right panels report the results for the full and before the COVID-19 pandemic out-of-sample periods, respectively. In addition to the portfolio allocation strategies described in Section 3, the out-of-sample performance of the Bovespa index was also included. Window sizes of $T = 60$ and $T = 120$ months yield to out-of-sample periods of 210 and 150 months in the full out-of-sample case and 182 and 122 months in the out-of-sample period before the COVID-19 pandemic, respectively.

Whichever the window size ($T = 60$ or $T = 120$) and the out-of-sample period considered (full or before the COVID-19 pandemic), the smallest SD is obtained by the MV portfolio strategy, closely followed by the HRP portfolio (for the full out-of-sample period) and by the MD portfolio (for the out-of-sample period before the COVID-19 pandemic). However, despite the differences observed (in absolute values), when the equality of variances is testing between HRP against each one of the other strategies, the bootstrap test of Ledoit and Wolf (2011) only rejects the null when HRP is tested against MV in all scenarios (p-value = 0.0056 for $T = 60$ and 0.0004 for $T = 120$ in the full out-of-sample period and p-value = 0.0424 for $T = 60$ and 0.0038 for $T = 120$ in the out-of-sample period before the COVID-19 pandemic) and when HRP is testing against MDE for $T = 120$ in the out-of-sample period before the COVID-19 pandemic (p-value = 0.0220). In all other cases, the null is not rejected at 5% of significance, meaning that, except in the aforementioned situations, no evidence of differences in the out-of-sample variance between HRP and its competitors is observed.

In terms of SR, the highest and smallest values are obtained by the MDE portfolio allocation strategy and by the market index (IBOV), respectively. The equality of SR for HRP against each one of the other strategies is tested using the bootstrap test of Ledoit and Wolf (2008), which rejects the null

[2]This value is quite conservative in the Brazilian scenario. Actually, a monthly risk-free rate of 0.5% is smaller than the average monthly Brazilian interest rate in the period analyzed.

when HRP is compared against IBOV in three out of the four scenarios (p-value = 0.0286 for $T = 60$ and 0.0420 for $T = 120$ in the full out-of-sample period and p-value = 0.0250 for $T = 120$ in the out-of-sample period before the COVID-19 pandemic). In all other cases, the null is not rejected at 5% of significance, indicating no evidence of differences in the SR between HRP and its competitors.

With respect to the two other risk-adjusted performance measures, namely, ASR and SO, results are similar to the obtained for the SR. An important difference is observed between the portfolio allocation strategies implemented concerning the impact of transaction costs in each one of them. Overall, HRP reports the highest average turnover, which means that this strategy is more impacted by transaction costs than all others. Additionally, although HRP does not report the worst performance in terms of diversification, it is always outperformed by the IV, RP, and EW portfolios.

In general, if risk-adjusted performance measures are of primary importance and transaction costs can be neglected, HRP performs as well as all other strategies.[3] However, because of its larger values in absolute terms, MDE could be preferred. If either transaction cost or diversification measures cannot be neglected, HRP is not a preferable strategy due to its high average turnover and SSPW. Conversely, if risk is of primary importance, MV portfolio should be preferred.

Figure 1 displays the accumulated returns over the full out-of-sample period of all strategies implemented and reported in Table 4 when $T = 120$. Note the poor performance of IBOV in comparison to all other strategies and also note that all strategies suddenly drop in March 2020, when the World Health Organization declared the SARS-COVID-19 virus as a pandemic.

A similar analysis considering only ordinary stocks and excluding high correlated assets of the same company was also performed.[4] The results are reported in Table 3 and in general are similar to those obtained in Table 2, yielding to the same conclusions.

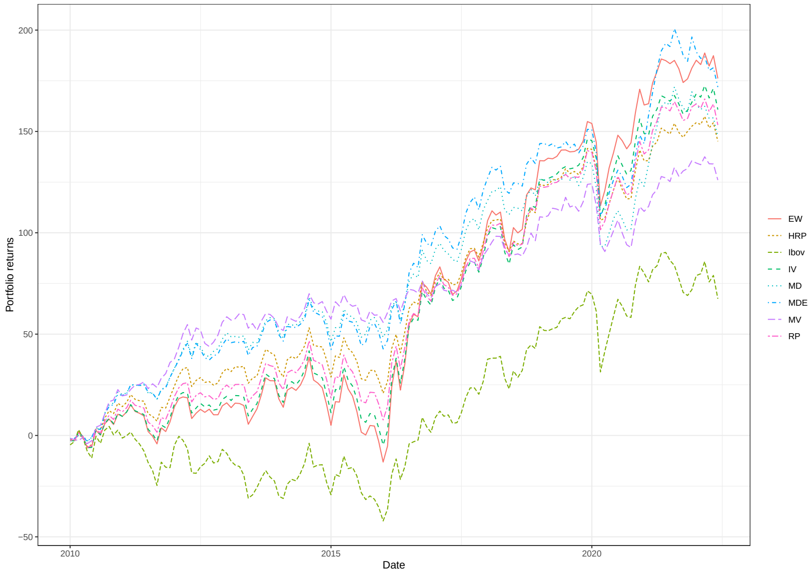
5. Conclusions

This paper has empirically evaluated the recently proposed HRP portfolio allocation strategy for the Brazilian financial market. The out-of-sample performance was compared against six widely-known portfolio allocation strate-

[3] For the SR, the bootstrap test of Ledoit and Wolf (2008) helps verify this. For the other two risk-adjusted measures, no bootstrap test is available to its use and one focused on the magnitude of the differences, which is almost similar to the obtained in the SR case.

[4] The assets removed were BBDC4, ELET6, PETR4, ITSA4 and GGBR4.

Figure 1
Accumulated out-of-sample portfolio returns from to January 2010 to June 2022



gies, as well as the Bovespa index. The comparison considered six out-of-sample performance measures and included different windows sizes and out-of-sample periods, yielding to four scenarios.

In terms of SD, the MV portfolio achieves the best performance and, according to the test of Ledoit and Wolf (2011), outperforms HRP. The same test does not reject the null of equality of variance of HRP against each one of the other strategies.

When the SR is of interest, HRP and several other strategies perform statistically equal, with no preference among them, except in case of transaction cost or diversification are important, HRP is not the best option. Similar results are obtained when the two other risk-adjusted performance measures are of interest.

In general terms, while HRP is not the best strategy it is not a strategy to be ruled out when portfolios of Brazilian assets are build, except possibly when transaction costs are of crucial importance.

Finally, clustering-based machine learning portfolio allocation strategies are a new, innovative idea to deal with diversification. There are many contributions to be developed in this direction, including other ouf-of-sample comparisons, the effect of covariance matrix misspecification on the portfolio weights, among others. Some of these points are in the research agenda of the third author.

Acknowledgments

The third author acknowledges financial support from grant 2022/09122-0, São Paulo Research Foundation (FAPESP) and Programa de Incentivo a Novos Docentes da UNICAMP (PIND) grant 2525/23. All authors acknowledge the computational support of the Center for Applied Research on Econometrics, Finance, and Statistics (CAREFS), Brazil.

References

Ardia, D., Bolliger, G., Boudt, K. and Gagnon-Fleury, J.-P. (2017). The impact of covariance misspecification in risk-based portfolios, *Annals of Operations Research* **254**: 1–16.

Ardia, D., Boudt, K. and Gagnon-Fleury, J.-P. (2017). Riskportfolios: Computation of risk-based portfolios in R, *Journal of Open Source Software* **10**(2).

Bai, X., Scheinberg, K. and Tutuncu, R. (2016). Least-squares approach to risk parity in portfolio selection, *Quantitative Finance* **16**(3): 357–376.

Battaglia, T. K. and Leal, R. P. (2017). Equally weighted portfolios of randomly selected stocks and the individual investor, *Latin American Business Review* **18**(1): 69–90.

Becker, F., Gürtler, M. and Hibbeln, M. (2015). Markowitz versus Michaud: Portfolio optimization strategies reconsidered, *The European Journal of Finance* **21**(4): 269–291.

Borges, B., Caldeira, J. F. and Ziegelmann, F. A. (2015). Selection of minimum variance portfolio using intraday data: An empirical comparison among different realized measures for bm&fbovespa data, *Brazilian Review of Econometrics* **35**(1): 23–46.

Bortoluzzo, M. M., Bortoluzzo, A. B., Imamura, R. A., Melhado, T. T., De, R. D. C. M. L. et al. (2018). Comparação do desempenho de carteiras utilizando ds métodos paridade de risco, mínima variância e equal weighting: Um estudo no mercado brasileiro em períodos pré, durante e pós a crise de 2008, *Revista Evidenciaçao Contábil & Finanças* **6**(3): 37–53.

Burggraf, T. (2021). Beyond risk parity–A machine learning-based hierarchical risk parity approach on cryptocurrencies, *Finance Research Letters* **38**: 101523.

Caldeira, J. F., Moura, G. V. and Santos, A. A. (2013). Seleção de carteiras utilizando o modelo Fama-French-Carhart, *Revista Brasileira de Economia* **67**: 45–65.

Caldeira, J. F., Moura, G. V., Perlin, M. S. and Santos, A. A. (2017). Portfolio management using realized covariances: Evidence from Brazil, *EconomiA* **18**(3): 328–343.

Chaves, D., Hsu, J., Li, F. and Shakernia, O. (2011). Risk parity portfolio vs. other asset allocation heuristic portfolios, *The Journal of Investing* **20**(1): 108–118.

Chopra, V. K., Hensel, C. R. and Turner, A. L. (1993). Massaging mean-variance inputs: Returns from alternative global investment strategies in the 1980s, *Management Science* **39**(7): 845–855.

Chopra, V. K. and Ziemba, W. T. (1993). The effect of errors in means, variances, and covariances on optimal portfolio choice, *The Journal of Portfolio Management* **19**(2): 6–11.

Choueifaty, Y., Froidure, T. and Reynier, J. (2013). Properties of the most diversified portfolio, *Journal of Investment Strategies* **2**(2): 49–70.

Choueifaty, Y. and Coignard, Y. (2008). Toward maximum diversification, *The Journal of Portfolio Management* **35**(1): 40–51.

Christoffersen, P., Errunza, V., Jacobs, K. and Langlois, H. (2012). Is the potential for international diversification disappearing? A dynamic copula approach, *The Review of Financial Studies* **25**(12): 3711–3751.

De Carvalho, R. L., Lu, X. and Moulin, P. (2012). Demystifying equity risk-based strategies: A simple alpha plus beta description, *The Journal of Portfolio Management* **38**(3): 56–70.

De Nard, G., Ledoit, O. and Wolf, M. (2021). Factor models for portfolio selection in large dimensions: The good, the better and the ugly, *Journal of Financial Econometrics* **19**(2): 236–257.

DeMiguel, V., Garlappi, L. and Uppal, R. (2009). Optimal versus naive diversification: How inefficient is the 1/N portfolio strategy?, *The Review of Financial Studies* **22**(5): 1915–1953.

du Plessis, H. and van Rensburg, P. (2020). Risk-based portfolio sensitivity to covariance estimation, *Investment Analysts Journal* **49**(3): 243–268.

Duarte, F. G. and De Castro, L. N. (2020). A framework to perform asset allocation based on partitional clustering, *IEEE Access* **8**: 110775–110788.

Duchin, R. and Levy, H. (2009). Markowitz versus the Talmudic portfolio diversification strategies, *The Journal of Portfolio Management* **35**(2): 71–74.

Engle, R. F., Ledoit, O. and Wolf, M. (2019). Large dynamic covariance matrices, *Journal of Business & Economic Statistics* **37**(2): 363–375.

Fan, J., Zhang, J. and Yu, K. (2012). Vast portfolio selection with gross-exposure constraints, *Journal of the American Statistical Association* **107**(498): 592–606.

Farias, C. A., Vieira, W. d. C. and Santos, M. L. d. (2006). Portfolio selection models: Comparative analysis and applications to the Brazilian stock market, *Revista de Economia e Agronegocio/Brazilian Review of Economics and Agribusiness* **4**(822-2016-54112): 387–407.

Freitas, W. B. and Junior, J. R. B. (2023). Random walk through a stock network and predictive analysis for portfolio optimization, *Expert Systems with Applications* p. 119597.

Fugazza, C., Guidolin, M. and Nicodano, G. (2015). Equally weighted vs. long-run optimal portfolios, *European Financial Management* **21**(4): 742–789.

Gambacciani, M. and Paolella, M. S. (2017). Robust normal mixtures for financial portfolio allocation, *Econometrics and Statistics* **3**: 91–111.

Goetzmann, W. N. and Kumar, A. (2008). Equity portfolio diversification, *Review of Finance* **12**(3): 433–463.

Goltz, F. and Sivasubramanian, S. (2018). Scientific beta maximum decorrelation indices, *Working Paper: ERI Scientific Beta Publication* .

Huang, M. and Yu, S. (2020). A new procedure for resampled portfolio with shrinkaged covariance matrix, *Journal of Applied Statistics* **47**(4): 642–652.

Iquiapaza, R. A., Vaz, G. F. C. and Borges, S. L. (2016). Portfolio evaluation of volatility timing and reward to risk timing investment strategies: The Brazilian case, *Revista de Finanças Aplicadas* **7**(2): 1–19.

Konno, H. and Yamazaki, H. (1991). Mean-absolute deviation portfolio optimization model and its applications to Tokyo stock market, *Management Science* **37**(5): 519–531.

Leal, R. P. C. and Campani, C. H. (2016). Valor-coppead indices, equally weighed and minimum variance portfolios, *Brazilian Review of Finance* **14**(1): 45–65.

Ledoit, O. and Wolf, M. (2008). Robust performance hypothesis testing with the Sharpe ratio, *Journal of Empirical Finance* **15**(5): 850–859.

Ledoit, O. and Wolf, M. (2011). Robust performances hypothesis testing with the variance, *Wilmott* **2011**(55): 86–89.

Lopez de Prado, M. (2016). Building diversified portfolios that outperform out-of-sample, *Journal of Portfolio Management* **42**(4): 59–69.

Maciel, L. (2021). A new approach to portfolio management in the Brazilian equity market: Does assets efficiency level improve performance?, *The Quarterly Review of Economics and Finance* **81**: 38–56.

Maillard, S., Roncalli, T. and Teïletche, J. (2010). The properties of equally weighted risk contribution portfolios, *The Journal of Portfolio Management* **36**(4): 60–70.

Malladi, R. and Fabozzi, F. J. (2017). Equal-weighted strategy: Why it outperforms value-weighted strategies? theory and evidence, *Journal of Asset Management* **18**: 188–208.

Markowitz, H. (1952). Portfolio selection, *Journal of Finance* **7**(1): 77–91.

Markowitz, H. (1959). *Portfolio selection: Efficient diversification of investments*, Vol. 16 of *Cowles Foundation Monograph*, 2 ed., Wiley, New York.

Merton, R. C. (1980). On estimating the expected return on the market: An exploratory investigation, *Journal of Financial Economics* **8**(4): 323–361.

Meucci, A. (2007). *Risk and asset allocation*, first ed., Springer, New York.

Michaud, R. and Michaud, R. (1998). *Efficient asset management: A practical guide to stock portfolio optimization and asset allocation*, first ed., Harvard Business School Press, Boston.

Michaud, R. O. (1989). The Markowitz optimization enigma: Is 'optimized' optimal?, *Financial Analysts Journal* **45**(1): 31–42.

Naibert, P. F. and Caldeira, J. F. (2015). Seleção de carteiras ótimas sob restrições nas normas dos vetores de alocação: Uma avaliação empírica com dados da BM&FBovespa, *Brazilian Review of Finance* **13**(3): 504–543.

Nakagawa, K., Imamura, M. and Yoshida, K. (2018). Risk-based portfolios with large dynamic covariance matrices, *International Journal of Financial Studies* **6**(2): 52.

Nourahmadi, M. and Sadeqi, H. (2022). A machine learning-based hierarchical risk parity approach: A case study of portfolio consisting of stocks of the top 30 companies on the Tehran stock exchange, *Financial Research Journal* **24**(2): 236–256.

Oliveira, A., Trucíos, C. and Valls Pereira, P. (2023). Does portfolio resampling really improve out-of-sample performance? Evidence from the Brazilian and US markets., *Working Paper* .

Pereira, P. L. V. and Oliveira, A. B. (2021). Estratégias de investimento em portfólios com estimativas de alta e baixa do mercado financeiro, *Brazilian Review of Finance* **19**(4): 160–185.

Pflug, G. C., Pichler, A. and Wozabal, D. (2012). The 1/N investment strategy is optimal under high model ambiguity, *Journal of Banking & Finance* **36**(2): 410–417.

Pézier, J. and White, A. (2008). The relative merits of alternative investments in passive portfolios, *The Journal of Alternative Investments* **10**(4): 37–49.

Qian, E. (2005). Risk parity portfolios: Efficient portfolios through true diversification, *Panagora Asset Management* .

Qian, E. (2006). On the financial interpretation of risk contribution: Risk budgets do add up, *Journal of Investment Management* **4**(4): 41–51.

Qian, E. (2011). Risk parity and diversification, *The Journal of Investing* **20**(1): 119–127.

Qian, E. (2013). Are risk-parity managers at risk parity?, *The Journal of Portfolio Management* **40**(1): 20–26.

Qian, E. (2016). *Risk parity fundamentals*, CRC Press.

Rubesam, A., Beltrame, A. L. et al. (2013). Minimum variance portfolios in the Brazilian equity market, *Brazilian Review of Finance* **11**(1): 81–118.

Santos, A. A. and Moura, G. V. (2014). Dynamic factor multivariate GARCH model, *Computational Statistics & Data Analysis* **76**: 606–617.

Santos, A. A. P. and Tessari, C. (2012). Técnicas quantitativas de otimização de carteiras aplicadas ao mercado de ações brasileiro, *Brazilian Review of Finance* **10**(3): 369–393.

Sen, J. and Dutta, A. (2022). A comparative study of hierarchical risk parity portfolio and eigen portfolio on the NIFTY 50 stocks, *Computational Intelligence and Data Analytics: Proceedings of ICCIDA 2022*, Springer, pp. 443–460.

Sen, J., Mehtab, S., Dutta, A. and Mondal, S. (2021). Hierarchical risk parity and minimum variance portfolio design on NIFTY 50 stocks, *2021 International Conference on Decision Aid Sciences and Application (DASA)*, IEEE, pp. 668–675.

Sharpe, W. F. (1975). Adjusting for risk in portfolio performance measurement, *The Journal of Portfolio Management* **1**(2): 29–34.

Simon, H. A. (1962). The architecture of complexity, *Proceedings of the American Philosophical Society* **106**(6): 467–482.

Sortino, F. A. and Van Der Meer, R. (1991). Downside risk, *Journal of Portfolio Management* **17**(4): 27.

Souza, L. C., Massardi, W. O., Pires, V. A. V. and Ciribeli, J. P. (2017). Otimização de carteira de investimentos: Um estudo com ativos do Ibovespa, *Revista de Gestão, Finanças e Contabilidade* **7**(3): 201–213.

Team, R. C. (2022). *R: A language and environment for statistical computing*, R Foundation for Statistical Computing, Vienna, Austria.

Trucios, C. (2021). *HierPortfolios: Hierarchical clustering-based portfolio allocation strategies*. R package version 0.1.0.

Trucíos, C., Mazzeu, J. H., Hallin, M., Hotta, L. K., Valls Pereira, P. L. and Zevallos, M. (2023). Forecasting conditional covariance matrices in high-dimensional time series: A general dynamic factor approach, *Journal of Business & Economic Statistics* **41**(1): 40–52.

Trucíos, C., Zevallos, M., Hotta, L. K. and Santos, A. A. (2019). Covariance prediction in large portfolio allocation, *Econometrics* **7**(2): 19.

Wolf, M. (2004). Resampling vs. shrinkage for benchmarked managers, *Shrinkage for Benchmarked Managers (July 2004)* .

Young, M. R. (1998). A minimax portfolio selection rule with linear programming solution, *Management Science* **44**(5): 673–683.

Table 2
Out-of-sample performance of portfolios

		Full out-of-sample period						Out-of-sample period before COVID					
	Method	SD	SR	ASR	SO	TO	SPPW	SD	SR	ASR	SO	TO	SPPW
$T = 60$	IBOV	23.24	0.19	0.19	0.26			21.91	0.25	0.25	0.34		
	EW	24.92	0.45	0.45	0.59	0.06	0.04	23.86	0.50	0.50	0.62	0.06	0.04
	MV	17.67	0.34	0.34	0.47	0.13	0.21	17.15	0.42	0.42	0.55	0.13	0.20
	RP	21.52	0.47	0.46	0.62	0.07	0.05	20.30	0.54	0.54	0.68	0.07	0.05
	MD	19.80	0.42	0.42	0.58	0.13	0.17	18.12	0.51	0.51	0.66	0.13	0.16
	MDE	21.75	0.49	0.48	0.65	0.14	0.14	19.70	0.58	0.58	0.72	0.14	0.14
	IV	22.85	0.44	0.44	0.58	0.06	0.04	21.85	0.50	0.51	0.63	0.06	0.04
	HRP	19.61	0.42	0.41	0.56	0.19	0.08	18.76	0.48	0.48	0.62	0.20	0.08
$T = 120$	IBOV	22.11	–0.03	–0.03	–0.04			19.71	0.00	0.00	0.00		
	EW	25.48	0.32	0.32	0.42	0.06	0.04	24.06	0.34	0.35	0.42	0.06	0.04
	MV	15.74	0.26	0.26	0.36	0.08	0.19	14.26	0.36	0.36	0.47	0.08	0.18
	RP	21.71	0.29	0.29	0.39	0.06	0.05	19.89	0.34	0.34	0.43	0.06	0.05
	MD	19.87	0.29	0.29	0.41	0.11	0.13	16.66	0.36	0.36	0.48	0.11	0.13
	MDE	21.84	0.36	0.35	0.48	0.11	0.12	18.53	0.42	0.42	0.53	0.11	0.12
	IV	23.32	0.29	0.29	0.39	0.06	0.04	21.93	0.34	0.34	0.42	0.06	0.04
	HRP	19.72	0.28	0.28	0.39	0.15	0.07	18.24	0.38	0.38	0.48	0.16	0.06

Portfolios were constructed using equal-weight, hierarchical risk parity and minimum variance portfolios. Full out-of-sample period (left panel) and out-of-sample period before COVID-19 pandemic (right panel)

Table 3
Out-of-sample performance of portfolios – ordinary stocks excluding same-company highly correlated assets

		Full out-of-sample period						Out-of-sample period before COVID					
	Method	SD	SR	ASR	SO	TO	SPPW	SD	SR	ASR	SO	TO	SPPW
$T = 60$	IBOV	23.24	0.19	0.19	0.26			21.91	0.25	0.25	0.34		
	EW	23.87	0.46	0.46	0.60	0.07	0.05	22.69	0.51	0.52	0.64	0.06	0.05
	MV	17.60	0.35	0.35	0.48	0.12	0.21	17.07	0.43	0.43	0.56	0.13	0.20
	RP	20.52	0.46	0.45	0.62	0.07	0.06	19.21	0.54	0.54	0.69	0.07	0.06
	MD	19.78	0.42	0.41	0.57	0.13	0.17	18.09	0.50	0.50	0.65	0.13	0.16
	MDE	21.74	0.48	0.47	0.64	0.13	0.14	19.68	0.57	0.57	0.71	0.13	0.14
	IV	21.58	0.45	0.44	0.59	0.06	0.06	20.50	0.52	0.52	0.65	0.06	0.06
	HRP	18.85	0.43	0.42	0.58	0.18	0.10	17.92	0.51	0.50	0.65	0.18	0.09
$T = 120$	IBOV	22.11	–0.03	–0.03	–0.04			19.71	0.00	0.00	0.00		
	EW	24.39	0.34	0.33	0.45	0.07	0.05	22.78	0.38	0.38	0.47	0.07	0.05
	MV	15.73	0.26	0.26	0.36	0.08	0.19	14.25	0.36	0.36	0.48	0.08	0.18
	RP	20.77	0.31	0.30	0.42	0.06	0.06	20.48	0.38	0.38	0.48	0.06	0.05
	MD	19.87	0.29	0.29	0.40	0.10	0.14	18.80	0.38	0.38	0.48	0.06	0.06
	MDE	21.85	0.35	0.35	0.47	0.10	0.12	16.66	0.36	0.36	0.47	0.10	0.13
	IV	21.99	0.32	0.32	0.43	0.06	0.05	18.54	0.41	0.41	0.52	0.10	0.12
	HRP	18.63	0.33	0.33	0.46	0.14	0.08	16.91	0.43	0.43	0.55	0.14	0.07

Portfolios were constructed using equal-weight, hierarchical risk parity and minimum variance portfolios. Full out-of-sample period (left panel) and out-of-sample period before COVID-19 pandemic (right panel)